

Bethe vectors for models based on the super-Yangian $Y(\mathfrak{gl}(m|n))$

S. Z. Pakuliak^{a,b}, E. Ragoucy^c, N. A. Slavnov^d ¹

^a *Moscow Institute of Physics and Technology, Dolgoprudny, Moscow reg., Russia*

^b *Laboratory of Theoretical Physics, JINR, Dubna, Moscow reg., Russia*

^c *Laboratoire de Physique Théorique LAPTH, CNRS and Université de Savoie,
BP 110, 74941 Annecy-le-Vieux Cedex, France*

^d *Steklov Mathematical Institute, Moscow, Russia*

Abstract

We study Bethe vectors of integrable models based on the super-Yangian $Y(\mathfrak{gl}(m|n))$. Starting from the super-trace formula, we exhibit recursion relations for these vectors in the case of $Y(\mathfrak{gl}(2|1))$ and $Y(\mathfrak{gl}(1|2))$. These recursion relations allow to get explicit expressions for the Bethe vectors. Using an antimorphism of the super-Yangian $Y(\mathfrak{gl}(m|n))$, we also construct a super-trace formula for dual Bethe vectors, and, for $Y(\mathfrak{gl}(2|1))$ and $Y(\mathfrak{gl}(1|2))$ super-Yangians, show recursion relations for them. Again, the latter allow us to get explicit expressions for dual Bethe vectors.

1 Introduction

Algebraic Bethe Ansatz (ABA) is a powerful tool for the calculation of Bethe vectors (BVs) of integrable models, which allows to access to the correlation functions of these models, through the calculation of the form factors (see e.g. [1, 2, 3, 4] and references therein). It has been successfully applied for models based on $\mathfrak{gl}(2)$ or its quantum deformation. In that case, one uses a determinant presentation for the norm and the scalar product of BVs [5, 6] to get an easy-to-handle expression for the form factors. The latter then can be used to study the thermodynamic limit of the form factors and get insight on the correlation functions asymptotic behavior [7, 8].

¹ stanislav.pakuliak@jinr.ru, eric.ragoucy@lapth.cnrs.fr, nslavnov@mi.ras.ru

However for models based on higher rank Lie algebras (or their quantum deformation), much less is known. Already at the level of BVs, although the ABA was performed long ago [9], few explicit expressions have been obtained, apart from the scalar product obtained in [10] that is difficult to handle. Recently, in a series of papers, the case of $\mathfrak{gl}(3)$ and of its quantum deformation was successfully studied, starting from explicit forms for BVs [11, 12] and the calculation of their scalar products [13, 14], up to determinant presentations for form factors [15]. The case of Bose gas with two internal degrees of freedom was also tackled, again with explicit expressions for Bethe vectors [16, 17], and determinant presentations of the form factors [18, 19, 20]. The case of more general Lie algebra (or their quantum deformation) remains to be done, but some steps have been done towards their resolution, using the current presentation of these algebras [21]. Note also that a trace formula for BVs is known for the $\mathfrak{gl}(n)$ [22] that can be used to deduce more properties of Bethe vectors.

The case of superalgebras is even less rich, apart from a super-trace formula in the $\mathfrak{gl}(m|n)$ case [23]. It is rather unfortunate, given their relevance in the study of gauge theories [24, 25, 26], in particular super-Yang-Mills (SYM) theories and AdS/CFT correspondence (see [27] and references therein). Indeed it is now believed that integrability should play an important role in SYM theories based on $PSL(4|4)$ [28], and also in its subsectors, such as $PSL(2|2)$ or $SL(2|1)$ [29]. Moreover, the t-J model, well-known in condensed matter physics, is based on the $\mathfrak{gl}(2|1)$ superalgebra [30]. Thus, there is some urge to find explicit representations for the Bethe vectors for integrable models based on these superalgebras. The aim of this paper is to present explicit expressions for Bethe vectors of integrable models based on $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(1|2)$.

The method we will be using mimics the one used for the $\mathfrak{gl}(3)$. Starting from the super-trace formula, we will deduce some recursion relations obeyed by the BVs. Then, solving these recursions, we will obtain explicit expressions for BVs. Using different morphisms, we will use the solution for the $Y(\mathfrak{gl}(2|1))$ case to construct solutions for the $Y(\mathfrak{gl}(1|2))$ case and also for the dual BVs.

The plan of our paper follows the lines we mentioned. After reminding some properties of Yangians $Y(\mathfrak{gl}(m|n))$, based on $\mathfrak{gl}(m|n)$ Lie superalgebras in section 2, we will remind the super-trace formula for the case of $Y(\mathfrak{gl}(2|1))$ and $Y(\mathfrak{gl}(1|2))$ in section 3. Then, using the super-trace formula, we will show in section 4 that the BVs obey a recursion relation. A second recursion relation will be exhibited in section 5. These recursion relations allow to get explicit formulas for BVs of $Y(\mathfrak{gl}(2|1))$ models (section 7) and then for $Y(\mathfrak{gl}(1|2))$ models and for dual BVs (section 8).

2 $\mathfrak{gl}(m|n)$ rational R -matrices

2.1 Graded vector spaces

We will work with graded vector spaces. We introduce the \mathbb{Z}_2 -grading

$$[\cdot] : \{1, 2, \dots, m+n\} \rightarrow \{0, 1\},$$

where $[j] = 0$ for m of the above integers, while $[j] = 1$ for the remaining n integers. The basic vector space will be \mathbb{C}^{m+n} , equipped with a gradation that we will loosely write $[\cdot]$:

$$[\cdot] : \begin{cases} \mathbb{C}^{m+n} & \rightarrow \mathbb{Z}_2 \\ e_a & \rightarrow [a], \end{cases}$$

so that the vector space will be noted $\mathbb{C}^{m|n}$. The elementary matrices of $\text{End}(\mathbb{C}^{m|n})$ will be graded accordingly $[E_{ij}] = [i] + [j]$. The grading is a morphism for the multiplication, so that

$$[E_{ij}E_{kl}] = [E_{ij}] + [E_{kl}] = [i] + [j] + [k] + [l].$$

Vectors or matrices that have \mathbb{Z}_2 -grading 0 (resp. 1) will be called even (resp. odd).

The gradation we will be mainly using has the form

$$[i] = \begin{cases} 0, & i = 1, 2, \dots, m, \\ 1, & i = m + 1, \dots, m + n. \end{cases} \quad (2.1)$$

We will call it the *distinguished gradation*, because it corresponds to the grading associated to the distinguished Dynkin diagram of $\mathfrak{gl}(m|n)$, with only one fermionic simple root. Other gradations can be defined, such as

$$\begin{cases} [2i - 1] = 1, \\ [2i] = 0, \end{cases} \quad i = 1, 2, \dots \quad (2.2)$$

which leads to a 'grey' Dynkin diagram where the simple roots are all fermionic. A third example is given by

$$\begin{cases} [4i + 1] = [4i + 2] = 1, \\ [4i + 3] = [4i + 4] = 0, \end{cases} \quad i = 0, 1, 2, \dots \quad (2.3)$$

which leads to an 'alternating' Dynkin diagram, where the simple roots are alternatively fermionic and bosonic. Hereafter, by $[\cdot]$ we will always understand the distinguished gradation given by (2.1).

On $\text{End}(\mathbb{C}^{m|n})$ we define a super-trace operator which is graded-cyclic:

$$\text{str}(E_{ij}) = (-1)^{[j]} \delta_{ij}, \quad \text{str}(E_{ij} E_{kl}) = (-1)^{([i]+[j])([k]+[l])} \text{str}(E_{kl} E_{ij}).$$

We also define a supertransposition

$$E_{ij}^t = (-1)^{[j][i]+[j]} E_{ji}. \quad (2.4)$$

The graded transposition is compatible with the super-trace: for any matrices A and B

$$\text{str} A = \text{str} A^t \quad \text{and} \quad \text{str}(A^t B^t) = \text{str}(A B). \quad (2.5)$$

Note that the graded transposition is idempotent¹ of order 4, not of order 2, a common feature in superalgebras. Indeed for a matrix A of given grade, we have

$$(A^t)^t = (-1)^{[A]} A. \quad (2.6)$$

¹One could define a transposition $(E^T)_{ij} = (-1)^{[i][j]} E_{ji}$, which is an antimorphism of order 2. However it does not obey relation (2.5).

Tensor products of $\mathbb{C}^{m|n}$ spaces will be also graded:

$$(\mathbb{I} \otimes E_{ij}) \cdot (E_{kl} \otimes \mathbb{I}) = (-1)^{([i]+[j])([k]+[l])} E_{kl} \otimes E_{ij}.$$

where $\mathbb{I} = \sum_{i=1}^{m+n} E_{ii}$ is a unit matrix.

Together with this grading of spaces and tensor products, one defines a graded permutation operator

$$P = \sum_{i,j=1}^{m+n} (-1)^{[j]} E_{ij} \otimes E_{ji}$$

which obeys

$$P^2 = \mathbb{I} \otimes \mathbb{I} \quad \text{and} \quad P^{t_1 t_2} = P.$$

It acts on tensor products of vectors and matrices as:

$$P e_i \otimes e_j = (-1)^{[i][j]} e_j \otimes e_i \quad \text{and} \quad P E_{ij} \otimes E_{kl} P = (-1)^{([i]+[j])([k]+[l])} E_{kl} \otimes E_{ij}.$$

2.2 Super-Yangian algebras $Y(\mathfrak{gl}(m|n)) \equiv Y(m|n)$

The super-Yangian $Y(\mathfrak{gl}(m|n)) \equiv Y(m|n)$ is an associative algebra with a unit element $\mathbf{1}$ generated by the modes $T_{ij}^{(p)}$, $p \in \mathbb{Z}_+$ of the *universal monodromy matrix*

$$T(u) = \sum_{i,j=1}^{m+n} E_{ij} \otimes T_{ij}(u) = \mathbb{I} \otimes \mathbf{1} + \sum_{i,j=1}^{m+n} \sum_{p=0}^{\infty} E_{ij} \otimes T_{ij}^{(p)} \left(\frac{c}{u} \right)^{p+1}, \quad (2.7)$$

defined by the *RTT* relation:

$$R_{12}(u_1, u_2) T_1(u_1) T_2(u_2) = T_2(u_2) T_1(u_1) R_{12}(u_1, u_2) \quad (2.8)$$

$$\text{with} \quad R_{12}(u_1, u_2) = \mathbb{I} + g(u_1, u_2) P_{12} \quad \text{and} \quad g(u, v) = \frac{c}{u - v}, \quad (2.9)$$

where c is a constant. We will loosely say that the series $T_{ij}(u)$ belongs to the super-Yangian, although strictly speaking they belong to $Y(m|n)[[u^{-1}]]$. The relation (2.8) is written in the tensor product $\text{End}(\mathbb{C}^{m|n}) \otimes \text{End}(\mathbb{C}^{m|n}) \otimes Y(m|n)[[u^{-1}]]$, and the indices indicate which copy of $\text{End}(\mathbb{C}^{m|n})$ the operators belong to.

The \mathbb{Z}_2 -grading $[\cdot]$ is extended to the super-Yangian through

$$[T_{ij}(u)] = [T_{ij}^{(p)}] = [i] + [j], \quad \forall u \in \mathbb{C}, \quad \forall p \in \mathbb{Z}_+.$$

As for matrices, generators that have \mathbb{Z}_2 -grading 0 (resp. 1) will be called even (resp. odd). Relations between these generators are given below, but let us first stress that, because of the graded tensor product, the order in the tensor product matters. Indeed, for instance $T(u) E_{kl} = \sum_{i=1}^{m+n} (-1)^{([i]+[k])([l]+[i])} E_{il} \otimes T_{ik}(u)$ while for $\tilde{T}(u) = \sum_{i,j=1}^{m+n} T_{ij}(u) \otimes E_{ij}$, one has $\tilde{T}(u) E_{kl} = \sum_{i=1}^{m+n} T_{ik}(u) \otimes E_{il}$.

The R -matrix (2.9) is unitary and symmetric:

$$R_{21}(u_2, u_1) R_{12}(u_1, u_2) = f(u_1, u_2) f(u_2, u_1) \mathbb{I} \otimes \mathbb{I}, \quad (2.10)$$

$$R_{21}(u_1, u_2) = P R_{12}(u_1, u_2) P = R_{12}(u_1, u_2) = R_{12}(u_1, u_2)^{t_1 t_2}, \quad (2.11)$$

where $f(u, v) = 1 + g(u, v)$.

Projecting the relation (2.8), one gets the commutation relations for the super-Yangian $Y(m|n)$:

$$[T_{ij}(z), T_{kl}(w)] = (-1)^{[l]([i]+[j])+[i][j]} g(z, w) (T_{il}(z) T_{kj}(w) - T_{il}(w) T_{kj}(z)), \quad (2.12)$$

where we have introduced the graded commutator

$$[T_{ij}(z), T_{kl}(w)] = T_{ij}(z) T_{kl}(w) - (-1)^{([i]+[j])([k]+[l])} T_{kl}(w) T_{ij}(z).$$

When $n = 0$ we recover the Yangian based on $\mathfrak{gl}(m)$.

Remark that the monodromy matrix $T(u)$ is globally even, because the \mathbb{Z}_2 -grading of E_{ij} is the same as the one of $T_{ij}(u)$. The same is true for the R -matrix.

Note that, by definition, the commutator is graded anti-symmetric:

$$[T_{ij}(z), T_{kl}(w)] = -(-1)^{([i]+[j])([k]+[l])} [T_{kl}(w), T_{ij}(z)], \quad (2.13)$$

which implies in particular that

$$[T_{ij}(z), T_{kl}(w)] = (-1)^{[i]([k]+[l])+[k][l]} g(z, w) (T_{kj}(w) T_{il}(z) - T_{kj}(z) T_{il}(w)). \quad (2.14)$$

The graded transfer matrix is defined as

$$t(w) = \text{str } T(w) = \sum_{j=1}^{m+n} (-1)^{[j]} T_{jj}(w). \quad (2.15)$$

It defines an integrable system, due to the relation $[t(z), t(w)] = 0$.

Evaluation map and subalgebras of $Y(m|n)$

The generators $T_{i,j}^{(0)}$, $i, j = 1, 2, \dots, m+n$, form a Lie superalgebra $\mathfrak{gl}(m|n)$ with commutation relations

$$[T_{ij}^{(0)}, T_{kl}^{(0)}] = (-1)^{[i]([k]+[l])+[k][l]} (T_{kj}^{(0)} \delta_{il} - \delta_{kj} T_{il}^{(0)}).$$

They act naturally on the monodromy matrix:

$$[T_{ij}^{(0)}, T_{kl}(z)] = (-1)^{[i]([k]+[l])+[k][l]} (T_{kj}(z) \delta_{il} - \delta_{kj} T_{il}(z)).$$

In fact, as for the Yangian $Y(\mathfrak{gl}(m))$, there exists an *evaluation* morphism from $Y(m|n)$ to the Lie superalgebra $\mathfrak{gl}(m|n)$:

$$ev : \begin{cases} Y(m|n) \rightarrow \mathfrak{gl}(m|n) \\ T(u) \rightarrow \mathbb{I} \otimes \mathbf{1} + \frac{c}{u} \sum_{i,j=1}^{m+n} E_{ij} \otimes \mathfrak{e}_{ij} \end{cases} \quad \text{where} \quad \mathfrak{e}_{ij} \in \mathfrak{gl}(m|n),$$

where the $\mathfrak{gl}(m|n)$ generators \mathfrak{e}_{ij} just corresponds to the so-called *zero modes* $T_{ij}^{(0)}$. The latter are a symmetry of the integrable model described by the transfer matrix: $[T_{ij}^{(0)}, t(z)] = 0$.

As far as subalgebras are concerned, the generators $T_{ij}(u)$, $1 \leq i, j \leq m$ (resp. $m+1 \leq i, j \leq m+n$) form a Yangian $Y(\mathfrak{gl}(m))$ (resp. $Y(\mathfrak{gl}(n))$) subalgebra of $Y(m|n)$. However, they are not Hopf-subalgebras.

Shorthand notation used in the paper

To make the presentation easier to read, we will use the following notation throughout the paper.

Sets of parameters will be noted with a bar, such as $\bar{u} = \{u_1, u_2, \dots, u_\ell\}$. Let us stress that these sets will be ordered, due to the \mathbb{Z}_2 -grading. Elements of these sets will have a latin index, e.g. u_j , while subsets will have, as a rule, a roman index, e.g. \bar{u}_I . These subsets will come as partitions of the original set, so that when saying \bar{u} is divided into \bar{u}_I and \bar{u}_{II} , it will mean $\bar{u}_I \cap \bar{u}_{II} = \emptyset$ and $\bar{u}_I \cup \bar{u}_{II} = \bar{u}$. There will be one exception to these rules: the subset $\bar{u}_j = \bar{u} \setminus \{u_j\}$.

When considering functions of one or two variables, as $\lambda(z)$ or $f(z, w)$, the notation $\lambda(\bar{u})$ will mean product of the $\lambda(u_j)$ for each element u_j in the set \bar{u} . In the same way, $f(\bar{u}_{II}, z)$ will mean product of the $f(u_j, z)$ for each element u_j in the subset \bar{u}_{II} , and $f(\bar{u}_{II}, \bar{v})$ will mean the double product of $f(u_j, v_k)$ factors for each element u_j in the subset \bar{u}_{II} and each element in the set \bar{v} . These rule will also apply to operators that commute at different values of the parameters.

2.3 Representations and morphisms of super-Yangians

Automorphisms of $Y(m|n)$

The R -matrix and the monodromy matrix being globally even, it is easy to show from (2.8) that

$$\psi : \begin{array}{l} T(u) \rightarrow T^t(u) \\ T_{ij}(u) \rightarrow (-1)^{[i][j]+[i]} T_{ji}(u) \end{array} \quad \text{is an antimorphism of } Y(m|n). \quad (2.16)$$

We call this antimorphism a *transposition map*. Let us point out the difference of sign factor between E_{ij}^t and $\psi(T_{ij}(u))$, see (2.4) and (2.16), which ensures that $\psi(T(u)) = T^t(u)$. One can also check directly that the commutation relations (2.12) are consistent with the relations (for A and B of definite grading)

$$\psi([A, B]) = -[\psi(A), \psi(B)] \quad \text{and} \quad \psi(AB) = (-1)^{[A][B]} \psi(B)\psi(A).$$

As a byproduct, (2.6) shows that the application **gr** which multiplies any generator by its \mathbb{Z}_2 -grade

$$\mathbf{gr} : T_{ij}(u) \rightarrow (-1)^{[i]+[j]} T_{ij}(u) \quad (2.17)$$

is an automorphism of the algebra, since $\mathbf{gr} = \psi \circ \psi$. Another way to see that **gr** is an automorphism is to realize that it corresponds to a conjugation

$$\text{Ad}_\omega : T(u) \rightarrow \omega T(u) \omega^{-1} \quad \text{with} \quad \omega = \sum_{k=1}^{m+n} (-1)^{[k]} E_{kk}.$$

To construct Bethe vectors we will consider the right representation of the super-Yangian $Y(m|n)$ generated by a *singular* vector Ω such that

$$T_{ii}(u)\Omega = \lambda_i(u)\Omega, \quad T_{ij}(u)\Omega = 0, \quad \text{for } i > j. \quad (2.18)$$

For the dual Bethe vectors we will apply the transposition map ψ (2.16) to obtain the left representations of super-Yangian $Y(m|n)$ generated by the vector $\Omega^\dagger = \psi(\Omega)$ such that

$$\Omega^\dagger T_{ii}(u) = \lambda_i(u)\Omega^\dagger, \quad \Omega^\dagger T_{ij}(u) = 0, \quad \text{for } i < j. \quad (2.19)$$

The transposition maps right highest weight representations into left lowest weight representations. In that case, the weights of the left- and right- representations will be the same.

Isomorphism between $Y(m|n)$ and $Y(n|m)$

We consider now a morphism between $Y(m|n)$ and $Y(n|m)$.

Proposition 2.1. *Let $T_{ij}(u)$ be the generators of the Yangian $Y(m|n)$, and $\tilde{T}_{ij}(u)$ be the generators of the Yangian $Y(n|m)$. Let $[\cdot]$ (resp. $\widetilde{[\cdot]}$) be the \mathbb{Z}_2 -grading of the $Y(m|n)$ super-Yangian (resp. $Y(n|m)$ super-Yangian). Then, the following mapping:*

$$\varphi : \begin{cases} Y(m|n) & \rightarrow & Y(n|m) \\ T_{ij}(u) & \rightarrow & (-1)^{[i][j]+[j]+1} \tilde{T}_{\bar{j}\bar{i}}(u) \\ [j] & \rightarrow & \widetilde{[j]} = [j] + 1 \end{cases} \quad (2.20)$$

where $\bar{j} = m+n+1-j$, defines an isomorphism between $Y(m|n)$ and $Y(n|m)$ which is compatible with the supertrace operation.

Proof: The grading in $Y(m|n)$ is given by $[j] = 0$ when $j \leq m$ and $[j] = 1$ when $j > m$, which can be reformulated as $\widetilde{[j]} = 0$ when $\bar{j} > n$ and $\widetilde{[j]} = 1$ when $\bar{j} \leq n$. Up to a global shift of 1 modulo 2, $\widetilde{[j]}$ in $Y(m|n)$ corresponds to $[j]$ in $Y(n|m)$. Then, by a direct calculation, starting from the relation (2.12) in $Y(m|n)$ one gets through φ the relation (2.14) for $Y(n|m)$.

In fact any multiplication by $(-1)^{[i]}$ and/or by $(-1)^{[j]}$ will keep the morphism property. We partially fix this freedom by demanding that the image of $str T(u)$ is $str \tilde{T}(u)$: there remain only two possibilities, one is given in (2.20), the other one is $T_{ij}(u) \rightarrow (-1)^{[i][j]+[i]+1} \tilde{T}_{\bar{j}\bar{i}}(u)$. ■

The isomorphism φ induces isomorphism between representations of the two super-Yangians. In that case, the highest weights map as

$$\varphi : (\lambda_1(u), \dots, \lambda_{m+n}(u)) \rightarrow (\tilde{\lambda}_1(u), \dots, \tilde{\lambda}_{m+n}(u)) \quad \text{with} \quad \tilde{\lambda}_{\bar{j}}(u) = -\lambda_{m+n+1-j}(u), \quad (2.21)$$

which just corresponds to the mapping of $T_{ij}(u)$ generators to $\tilde{T}_{\bar{j}\bar{i}}(u)$ ones.

Compositions of morphisms

The different morphisms can be composed, and we get relations among them. We focus on the morphisms φ , ψ and \mathbf{gr} , and to clarify the presentation, we fix m and n and call $T_{ij}(u)$ (resp. $\tilde{T}_{ij}(u)$) the elements of $Y(m|n)$ (resp. $Y(n|m)$). In the same way, we use the following notation:

$$\begin{array}{ccc} Y(m|n) & \xrightarrow{\varphi} & Y(n|m) \\ \psi \downarrow \mathbf{gr} & & \mathbf{gr} \downarrow \tilde{\psi} \\ Y(m|n) & \xleftarrow{\tilde{\varphi}} & Y(n|m) \end{array} \quad (2.22)$$

In the above diagram, we remind that $\varphi, \tilde{\varphi}, \mathbf{gr}$ and \mathbf{gr} are isomorphisms, while ψ and $\tilde{\psi}$ are antimorphisms, see section 2.3. Then, we have the following.

Lemma 2.1. *We have the following composition rules*

$$\tilde{\varphi} \circ \varphi = \text{id} \quad ; \quad \varphi \circ \tilde{\varphi} = \text{id} \quad ; \quad \psi \circ \psi = \mathbf{gr} \quad ; \quad \tilde{\psi} \circ \tilde{\psi} = \widetilde{\mathbf{gr}} \quad (2.23)$$

$$\tilde{\psi} \circ \varphi = \varphi \circ \psi \quad ; \quad \psi \circ \tilde{\varphi} = \tilde{\varphi} \circ \tilde{\psi} \quad ; \quad \widetilde{\mathbf{gr}} \circ \varphi = \varphi \circ \mathbf{gr} \quad ; \quad \mathbf{gr} \circ \tilde{\varphi} = \tilde{\varphi} \circ \widetilde{\mathbf{gr}}. \quad (2.24)$$

These rules imply in particular that the diagram (2.22) is commutative.

Proof: Direct calculation, applying the definitions (2.16), (2.20) and (2.17) to $T_{ij}(u)$ or $\tilde{T}_{ij}(u)$. ■

3 Bethe vectors

3.1 Super-trace formula

The generalization of the trace formula for Bethe vectors (introduced by Tarasov and Varchenko [22]) was given in [23] for super-Yangians and quantum deformations of super-symmetric affine algebras. In the case of super-Yangians, and specializing to the case of $Y(2|1)$ to simplify the presentation, it reads

$$\begin{aligned} \Phi_{a,b}(\bar{u}, \bar{v}) &= \frac{(-1)^{b(b+1)/2}}{H(\bar{v})} \text{str}_{1\dots a+b} \left[T_1(u_1) \cdots T_a(u_a) T_{a+1}(v_1) \cdots T_{a+b}(v_b) \mathbb{R}_{1\dots a+b} \right. \\ &\quad \left. \times E_{32}^{(a+b)} \cdots E_{32}^{(a+1)} E_{21}^{(a)} \cdots E_{21}^{(1)} \right] \Omega \end{aligned} \quad (3.1)$$

$$\begin{aligned} &= \frac{(-1)^b}{H(\bar{v})} \text{str}_{1\dots a+b} \left[T_1(u_1) \cdots T_a(u_a) T_{a+1}(v_1) \cdots T_{a+b}(v_b) \mathbb{R}_{1\dots a+b} \right. \\ &\quad \left. \times E_{21}^{(1)} \cdots E_{21}^{(a)} E_{32}^{(a+1)} \cdots E_{32}^{(a+b)} \right] \Omega, \end{aligned} \quad (3.2)$$

$$H(\bar{v}) = \prod_{1 \leq k < j \leq b} h(v_j, v_k) \quad \text{with} \quad h(u, v) = \frac{f(u, v)}{g(u, v)} = \frac{u - v + c}{c}. \quad (3.3)$$

where the super-trace str is taken over $(a+b)$ copies of the auxiliary space $\mathbb{C}^{2|1}$, a graded version of \mathbb{C}^3 , and $E_{jk}^{(p)}$ is the elementary matrix E_{jk} in the p^{th} copy of the auxiliary space. $\mathbb{R}_{1\dots a+b}$ is the following product of R -matrices:

$$\begin{aligned} \mathbb{R}_{1\dots a+b} &= (R_{a+1,a} \cdots R_{a+1,1})(R_{a+2,a} \cdots R_{a+2,1}) \cdots (R_{a+b,a} \cdots R_{a+b,1}) \\ &= (R_{a+1,a} \cdots R_{a+b,a})(R_{a+1,a-1} \cdots R_{a+b,a-1}) \cdots (R_{a+1,1} \cdots R_{a+b,1}) \end{aligned} \quad (3.4)$$

where we abbreviated $R_{a+j,k} \equiv R_{a+j,k}(v_j, u_k)$.

Note that the indices a, b in $\Phi_{a,b}(\bar{u}, \bar{v})$ indicate that $\#\bar{u} = a$ and $\#\bar{v} = b$.

Let us stress that since the tensor space is graded, the order of the elementary matrices $E_{32}^{(p)}$ matters, as illustrated in (3.1)-(3.2). In what follows, unless explicitly written, we will use the 'natural order', namely $(E_{32})^{\otimes b}$ stands for $E_{32}^{(a+1)} \cdots E_{32}^{(a+b)}$. The coefficient $(-1)^b/H(\bar{v})$ in (3.2) is for later convenience, see proposition 3.2.

Ω is the highest weight of the representation. It obeys (2.18) with

$$\lambda_j(z) = 1 + \frac{c}{z} \lambda_j^{(0)} + o(z^{-2}). \quad (3.5)$$

We give here some simple examples extracted directly from the super-trace formula:

$$\Phi_{a0}(\bar{u}, \emptyset) = T_{12}(u_1) \cdots T_{12}(u_a) \Omega \quad (3.6)$$

$$\Phi_{0,b}(\emptyset, \bar{v}) = \frac{1}{H(\bar{v})} T_{23}(v_1) \cdots T_{23}(v_b) \Omega \quad (3.7)$$

$$\Phi_{11}(u, v) = \left(T_{12}(u) T_{23}(v) + \lambda_2(v) g(v, u) T_{13}(u) \right) \Omega \quad (3.8)$$

$$\Phi_{12}(u, \{v_1, v_2\}) = h(v_2, v_1)^{-1} T_{12}(u) T_{23}(v_1) T_{23}(v_2) \Omega \quad (3.9)$$

$$+ g(v_2, v_1) T_{13}(u) \left(\lambda_2(v_1) g(v_1, u) T_{23}(v_2) - \lambda_2(v_2) g(v_2, u) T_{23}(v_1) \right) \Omega$$

$$\Phi_{21}(\bar{u}, v) = T_{12}(u_1) \Phi_{11}(u_2, v) + \lambda_2(v) g(v, u_1) f(v, u_2) T_{13}(u_1) \Phi_{1,0}(u_2, \emptyset). \quad (3.10)$$

The following property has been proven in [23] for the super-Yangian $Y(m|n)$. Again, to simplify the presentation, we just reproduce it for the two specific cases $Y(2|1)$ and $Y(1|2)$.

Proposition 3.1. *The BVs $\Phi_{a,b}(\bar{u}, \bar{v})$, as defined in (3.2) for the super-Yangian $Y(2|1)$ or in (4.12) for the super-Yangian $Y(1|2)$ are eigenvectors of the zero-modes $T_{jj}^{(0)}$, $j = 1, 2, 3$:*

$$T_{11}^{(0)} \Phi_{a,b}(\bar{u}, \bar{v}) = (\lambda_1^{(0)} - (-1)^{[1]} a) \Phi_{a,b}(\bar{u}, \bar{v}), \quad (3.11)$$

$$T_{22}^{(0)} \Phi_{a,b}(\bar{u}, \bar{v}) = (\lambda_2^{(0)} + (-1)^{[2]} (a - b)) \Phi_{a,b}(\bar{u}, \bar{v}), \quad (3.12)$$

$$T_{33}^{(0)} \Phi_{a,b}(\bar{u}, \bar{v}) = (\lambda_3^{(0)} + (-1)^{[3]} b) \Phi_{a,b}(\bar{u}, \bar{v}). \quad (3.13)$$

Moreover, if the Bethe equations

$$\frac{\lambda_2(u_j)}{\lambda_1(u_j)} = \frac{f_{[1]}(\bar{u}_j, u_j)}{f_{[2]}(u_j, \bar{u}_j)} \frac{1}{f_{[2]}(\bar{v}, u_j)}, \quad j = 1, 2, \dots, a, \quad (3.14)$$

$$\frac{\lambda_3(v_j)}{\lambda_2(v_j)} = f_{[2]}(v_j, \bar{u}) \frac{f_{[2]}(\bar{v}_j, v_j)}{f_{[3]}(v_j, \bar{v}_j)}, \quad j = 1, 2, \dots, b, \quad (3.15)$$

are obeyed, the BVs $\Phi_{a,b}(\bar{u}, \bar{v})$ are eigenvectors of the transfer matrix (2.15):

$$t(z) \Phi_{ab}(\bar{u}, \bar{v}) = \tau(z|\bar{u}, \bar{v}) \Phi_{ab}(\bar{u}, \bar{v}), \quad (3.16)$$

$$\begin{aligned} \tau(z|\bar{u}, \bar{v}) = & (-1)^{[1]} \lambda_1(z) f_{[1]}(\bar{u}, z) + (-1)^{[2]} \lambda_2(z) f_{[2]}(z, \bar{u}) f_{[2]}(\bar{v}, z) \\ & + (-1)^{[3]} \lambda_3(z) f_{[3]}(z, \bar{v}). \end{aligned} \quad (3.17)$$

The functions $f_{[j]}(u, v)$ are defined as

$$f_0(u, v) = 1 + g(u, v) = f(u, v) \quad \text{and} \quad f_1(u, v) = 1 - g(u, v) = f(v, u). \quad (3.18)$$

Note that due to the grading, in the $Y(2|1)$ case (resp. $Y(1|2)$ case), the ratio of f functions in the r.h.s. of the second (resp. first) Bethe equation cancels, and we are left with a free fermion equation.

Using the super-trace formula, one can show the following symmetry property

Proposition 3.2. *Let $\sigma_j = \sigma_{j,j+1}$ be the transposition between j and $j+1$. Then, for the BVs of $Y(2|1)$, we have*

$$\Phi_{a,b}(\bar{u}, \bar{v}) = \Phi_{a,b}(\bar{u}^{\sigma_j}, \bar{v}) \quad \text{and} \quad \Phi_{a,b}(\bar{u}, \bar{v}) = \Phi_{a,b}(\bar{u}, \bar{v}^{\sigma_j}). \quad (3.19)$$

In the case of $Y(m|n)$, denoting $\bar{t}^{(1)}, \dots, \bar{t}^{(m+n-1)}$ the sets of Bethe parameters, relation (3.19) will apply to the any set $\bar{t}^{(j)}$ $j = 1, 2, \dots, m+n-1$, provided one normalizes the super-trace with $H(\bar{t}^{(m)})$.

Proof: We prove the property for σ_{12} and $Y(2|1)$, but it extends trivially to σ_j and $Y(m|n)$. We first write $T_1(u_1)T_2(u_2) = R_{12}^{-1} T_2(u_2)T_1(u_1)R_{12}$. Then, iterative use of the Yang–Baxter equation shows that

$$R_{12} \mathbb{R}_{123\dots a+b} = \mathbb{R}_{213\dots a+b} R_{12}. \quad (3.20)$$

From cyclicity of the super-trace, one gets

$$\begin{aligned} \Phi_{a,b}(\bar{u}, \bar{v}) &= (-1)^{b(b+1)/2} \text{str}_{1\dots a+b} \left[T_2(u_2)T_1(u_1)T_3(u_3) \cdots T_a(u_a)T_{a+1}(v_1) \cdots T_{a+b}(v_b) \mathbb{R}_{213\dots a+b} \right. \\ &\quad \left. \times E_{32}^{(a+b)} \cdots E_{32}^{(a+1)} E_{21}^{(a)} \cdots E_{21}^{(3)} R_{12} E_{21}^{(2)} E_{21}^{(1)} R_{12}^{-1} \right] \Omega. \end{aligned} \quad (3.21)$$

Finally a direct calculation, using the explicit form of the R -matrix, shows that

$$R_{12} E_{21}^{(2)} E_{21}^{(1)} R_{12}^{-1} = E_{21}^{(1)} E_{21}^{(2)}. \quad (3.22)$$

Then, after relabeling of the auxiliary spaces 1 and 2, one recognizes in the right-hand-side $\Phi_{a,b}(\bar{u}^{t_1}, \bar{v})$, which proves the first relation of proposition 3.2.

The proof of the second relation follows the same lines, the only difference lies in the grade of E_{32} , which leads to

$$R_{a+1,a+2} E_{32}^{(a+2)} E_{32}^{(a+1)} R_{a+1,a+2}^{-1} = -\frac{f(v_2, v_1)}{f(v_1, v_2)} E_{32}^{(a+1)} E_{32}^{(a+2)} = \frac{h(v_2, v_1)}{h(v_1, v_2)} E_{32}^{(a+1)} E_{32}^{(a+2)}. \quad (3.23)$$

This coefficient cancels the one coming from the normalization factor $H(\bar{v})$. ■

3.2 Comparison with Tarasov–Varchenko trace formula

Starting from [22], and generalizing to the superalgebra case, one would get Bethe vectors defined as²

$$\mathcal{W}_{ab}(\bar{u}, \bar{v}) = \text{str} \left(T_1(u_1) \cdots T_a(u_a) T_{a+1}(v_1) \cdots T_{a+b}(v_b) \mathcal{R}_{1\dots a+b} (E_{21})^{\otimes a} (E_{32})^{\otimes b} \right) \quad (3.24)$$

with

$$\mathcal{R}_{1\dots a+b} = \prod_{1 \leq i < j \leq a+b}^{\rightarrow} R_{ji} = \left(\prod_{a+1 \leq i < j \leq a+b}^{\rightarrow} R_{ji} \right) \widetilde{\mathbb{R}}_{1\dots a+b} \left(\prod_{1 \leq i < j \leq a}^{\rightarrow} R_{ji} \right) \quad (3.25)$$

$$\widetilde{\mathbb{R}}_{1\dots a+b} = (R_{a+b,a} \cdots R_{a+b,1}) \cdots (R_{a+2,a} \cdots R_{a+2,1}) (R_{a+1,a} \cdots R_{a+1,1}) \quad (3.26)$$

$$\equiv \mathbb{R}_{1\dots a, a+b \dots a+1} \quad (3.27)$$

²However, be careful that the normalisation of the R -matrix is different in this paper.

where $\mathbb{R}_{1\dots a+b}$ is defined in (3.4). The form of $\mathcal{W}_{ab}(\bar{u}, \bar{v})$ is different from the one of $\Phi_{ab}(\bar{u}, \bar{v})$, given in [23], and reproduced in (3.1). However, we have the following.

Proposition 3.3. *In $Y(2|1)$, we have*

$$\mathcal{W}_{ab}(\bar{u}, \bar{v}) = \prod_{1 \leq i < j \leq a} f(u_j, u_i) \prod_{1 \leq i < j \leq b} g(v_j, v_i) \Phi_{ab}(\bar{u}^*, \bar{v}^*) \quad (3.28)$$

where for any set $\bar{w} = \{w_1, w_2, \dots, w_b\}$, we introduced the conjugate one $\bar{w}^* = \{w_b, \dots, w_2, w_1\}$.

Relation (3.28) trivially generalizes to the $Y(m|n)$ case.

Proof: A direct calculation shows that

$$\left(\prod_{1 \leq i < j \leq a}^{\rightarrow} R_{ji} \right) (\mathbb{E}_{21})^{\otimes a} = \prod_{1 \leq i < j \leq a} f(u_j, u_i) (\mathbb{E}_{21})^{\otimes a}.$$

Now, from the relation

$$T_{a+1}(v_1) \dots T_{a+b}(v_b) \left(\prod_{a+1 \leq i < j \leq a+b}^{\rightarrow} R_{ji} \right) = \left(\prod_{a+1 \leq i < j \leq a+b}^{\rightarrow} R_{ji} \right) T_{a+b}(v_b) \dots T_{a+1}(v_1),$$

the cyclicity property of the super-trace and the property

$$(\mathbb{E}_{32})^{\otimes b} \left(\prod_{a+1 \leq i < j \leq a+b}^{\rightarrow} R_{ji} \right) = \prod_{1 \leq i < j \leq b} f(v_j, v_i) (\mathbb{E}_{32})^{\otimes b}$$

we get

$$\begin{aligned} \mathcal{W}_{ab}(\bar{u}, \bar{v}) &= \prod_{1 \leq i < j \leq a} f(u_j, u_i) \prod_{1 \leq i < j \leq b} f(v_j, v_i) \\ &\times \text{str} \left(T_1(u_1) \dots T_a(u_a) T_{a+b}(v_b) \dots T_{a+1}(v_1) \tilde{\mathbb{R}}_{1\dots a+b} (\mathbb{E}_{21})^{\otimes a} (\mathbb{E}_{32})^{\otimes b} \right). \end{aligned} \quad (3.29)$$

Finally, one relabels the spaces $a+1, \dots, a+b$ into $a+b, \dots, a+1$ and the Bethe parameters v_j accordingly. One recovers the formula (3.28) after division by $H(\bar{v})$. \blacksquare

4 First recursion formula for $Y(2|1)$ and $Y(1|2)$ BVs

4.1 $Y(2|1)$ case

We focus now on the case of $Y(2|1)$. Recall that the grading is given by $[1] = [2] = 0$ and $[3] = 1$.

Proposition 4.1. *The $Y(2|1)$ Bethe vectors obey the following recursion relation:*

$$\begin{aligned} \Phi_{ab}(\bar{u}, \bar{v}) &= T_{12}(u_1) \Phi_{a-1,b}(\bar{u}_1, \bar{v}) \\ &+ \sum_{j=1}^b \lambda_2(v_j) g(v_j, u_1) f(v_j, \bar{u}_1) g(\bar{v}_j, v_j) T_{13}(u_1) \Phi_{a-1,b-1}(\bar{u}_1, \bar{v}_j), \end{aligned} \quad (4.1)$$

and we recall that $\bar{u}_1 = \bar{u} \setminus \{u_1\}$.

Proof: One extracts the space 1 dependence from the super-trace formula. To do it, we need the following formulas:

$$R_{21}(v, u) E_{32}^{(2)} E_{21}^{(1)} = E_{32}^{(2)} E_{21}^{(1)} + g(v, u) E_{31}^{(1)} E_{22}^{(2)} \quad \text{and} \quad R_{21}(v, u) E_{32}^{(2)} E_{31}^{(1)} = f(u, v) E_{32}^{(2)} E_{31}^{(1)}. \quad (4.2)$$

Applying these relations recursively on the matrices $R_{a+j,1}$, $j = 1, 2, \dots, b$, we get

$$\begin{aligned} H(\bar{v}) \Phi_{ab}(\bar{u}, \bar{v}) &= T_{12}(u_1) \Phi_{a-1,b}(\bar{u}_1, \bar{v}) - T_{13}(u_1) \sum_{j=1}^b (-1)^{b(b-1)/2+j} g(v_j, u_1) \prod_{1 \leq p < j} f(u_1, v_p) \\ &\times \text{str}_{2 \dots a+b} \left[T_2(u_2) \cdots T_{a+b}(v_b) \mathbb{R}_{2 \dots a+b} E_{32}^{(a+b)} \cdots E_{22}^{(a+j)} \cdots E_{32}^{(a+1)} E_{21}^{(a)} \cdots E_{21}^{(2)} \right] \Omega \end{aligned} \quad (4.3)$$

where we have used $\mathbb{R}_{1 \dots a+b} = \mathbb{R}_{2 \dots a+b} R_{a+1,1} \cdots R_{a+b,1}$.

It remains to eliminate the generator $E_{22}^{(a+j)}$, in the second term of eq. (4.3), which is done by performing the super-trace on the space $a+j$. For such a purpose, one first notes that

$$R_{a+j,k}(v, u) E_{22}^{(a+j)} E_{21}^{(k)} = (1 + g(v, u)) E_{22}^{(a+j)} E_{21}^{(k)}. \quad (4.4)$$

This shows that the elimination of the generator $E_{22}^{(a+j)}$, will produce a generator T_{22} . To get rid of this generator, one needs to move it to the right towards Ω , to get a λ_2 function. Moving $T_{22}(v_j)$ to the right is done thanks to the commutation relations³

$$T_{22}(x) T_{2k}(y) = g(x, y) T_{2k}(x) T_{22}(y) + f(y, x) T_{2k}(y) T_{22}(x). \quad (4.5)$$

Altogether, this implies that eliminating the generator $E_{22}^{(a+j)}$ (and summation on $j = 1, \dots, b$), we will get a sum $\sum_{\ell=1}^b \lambda_2(v_\ell)(\dots)$. To compute precisely the form of these terms, we will use the symmetry property 3.2, see below.

First, we compute the term corresponding to $\lambda_2(v_1)$. Because of the relations (4.5) and the order in the product of T 's in (4.3), it is clear that $\lambda_2(v_1)$ can be obtained only through $T_{22}(v_1)$, which in turn means that we need to determine how to get $E_{22}^{(a+1)}$ after $E_{22}^{(a+j)}$, $j = 1, \dots, b$, goes through $\mathbb{R}_{2 \dots a+b}$.

We fix j , and look at $E_{22}^{(a+j)}$. From the relations

$$R_{a+p,q} E_{32}^{(a+p)} E_{21}^{(q)} = E_{32}^{(a+p)} E_{21}^{(q)} + g(v_p, u_q) E_{22}^{(a+p)} E_{31}^{(q)}, \quad 1 \leq q \leq a; j < p \leq b \quad (4.6)$$

$$R_{a+j,k} E_{22}^{(a+j)} E_{21}^{(k)} = f(v_j, u_k) E_{22}^{(a+j)} E_{21}^{(k)}, \quad 1 \leq q \leq a \quad (4.7)$$

$$R_{a+p,q} E_{22}^{(a+p)} E_{31}^{(q)} = E_{22}^{(a+p)} E_{31}^{(q)} + g(v_p, u_q) E_{32}^{(a+p)} E_{21}^{(q)}, \quad 1 \leq q \leq a; j < p \leq b \quad (4.8)$$

it is clear that $E_{22}^{(a+j)}$ produces terms $E_{22}^{(a+p)}$ with $p \geq j$ only. It implies that there is only one way to get $E_{22}^{(a+1)}$ after going through $\mathbb{R}_{2 \dots a+b}$, and we get

$$f(v_1, \bar{u}_1) \text{str}_{2 \dots a+b} T_2(u_2) \cdots T_{a+b}(v_b) E_{22}^{(a+1)} \mathbb{R}_{2 \dots a, a+2 \dots a+b} E_{32}^{(a+b)} \cdots E_{32}^{(a+2)} E_{21}^{(a)} \cdots E_{21}^{(2)} \Omega. \quad (4.9)$$

³Because of the super-trace in space $p > a+j$ with the generator E_{32} , one knows by cyclicity that in $T_p(v)$ only the generators $T_{2k}(v)$ will matter.

It remains to perform the super-trace in space $a + 1$ and move $T_{22}(v_1)$ towards Ω . We obtain

$$\begin{aligned}\Phi_{ab}(\bar{u}, \bar{v}) &= T_{12}(u_1) \Phi_{a-1,b}(\bar{u}_1, \bar{v}) \\ &+ T_{13}(u_1) \lambda_2(v_1) g(v_1, u_1) f(v_1, \bar{u}_1) f(\bar{v}_1, v_1) \frac{H(\bar{v}_1)}{H(\bar{v})} \Phi_{a-1,b-1}(\bar{u}_1, \bar{v}_1) \\ &+ \sum_{\ell=2}^b \lambda_2(v_\ell)(\dots),\end{aligned}\tag{4.10}$$

where the dots encode all the terms that contribute to obtain $T_{22}(v_\ell)$ on the right.

In the same way, for $\bar{v}^\sigma = \{v_j, \bar{v}_j\}$, we have

$$\begin{aligned}\Phi_{ab}(\bar{u}, \bar{v}^\sigma) &= T_{12}(u_1) \Phi_{a-1,b}(\bar{u}_1, \bar{v}^\sigma) \\ &+ T_{13}(u_1) \lambda_2(v_j) g(v_j, u_1) f(v_j, \bar{u}_1) f(\bar{v}_j, v_j) \frac{H(\bar{v}_j)}{H(\bar{v}^\sigma)} \Phi_{a-1,b-1}(\bar{u}_1, \bar{v}_j) \\ &+ \sum_{\ell \neq j} \lambda_2(v_\ell)(\dots).\end{aligned}\tag{4.11}$$

But \bar{v}^σ is deduced from \bar{v} from action of the permutation $\sigma = \sigma_{12} \sigma_{23} \dots \sigma_{j-2,j-1} \sigma_{j-1,j}$ so that from property 3.2, $\Phi_{ab}(\bar{u}, \bar{v}^\sigma) = \Phi_{ab}(\bar{u}, \bar{v})$. It remains to compute

$$\frac{H(\bar{v}_j)}{H(\bar{v}^\sigma)} = h(\bar{v}_j, v_j)$$

to get the coefficient of $\lambda_2(v_j)$ in the recursion relation. ■

4.2 $Y(1|2)$ case

We focus now on the case of $Y(1|2)$. The grading is given by $\widetilde{[1]} = 0$ and $\widetilde{[2]} = \widetilde{[3]} = 1$. The super-trace formula read

$$\begin{aligned}\widetilde{\Phi}_{a,b}(\bar{u}, \bar{v}) &= \frac{(-1)^a}{H(\bar{u})} \text{str}_{1\dots a+b} \left[\widetilde{T}_1(u_1) \dots \widetilde{T}_a(u_a) \widetilde{T}_{a+1}(v_1) \dots \widetilde{T}_{a+b}(v_b) \mathbb{R}_{1\dots a+b} \right. \\ &\quad \left. \times E_{21}^{(1)} \dots E_{21}^{(a)} E_{32}^{(a+1)} \dots E_{32}^{(a+b)} \right] \Omega,\end{aligned}\tag{4.12}$$

where now E_{21} is odd while E_{32} is even. We have put a tilde on BVs to distinguish the $Y(1|2)$ BVs from the $Y(2|1)$ ones.

Proposition 4.2. *The $Y(1|2)$ Bethe vectors obey the following recursion relation:*

$$\begin{aligned}\widetilde{\Phi}_{ab}(\bar{u}, \bar{v}) &= h(\bar{u}_1, u_1)^{-1} \widetilde{T}_{12}(u_1) \widetilde{\Phi}_{a-1,b}(\bar{u}_1, \bar{v}) \\ &- h(\bar{u}_1, u_1)^{-1} \sum_{j=1}^b \widetilde{\lambda}_2(v_j) g(u_1, v_j) f(\bar{u}_1, v_j) f(\bar{v}_j, v_j) \widetilde{T}_{13}(u_1) \widetilde{\Phi}_{a-1,b-1}(\bar{u}_1, \bar{v}_j).\end{aligned}\tag{4.13}$$

Proof: The proof goes along the same line as for proposition 4.1, taking into account the difference between the gradings and the normalisation factor, which now depends on \bar{u} . For instance, the exchange relation (4.5) now reads

$$\tilde{T}_{22}(x) \tilde{T}_{2k}(y) = g(y, x) \tilde{T}_{2k}(x) \tilde{T}_{22}(y) + f(x, y) \tilde{T}_{2k}(y) \tilde{T}_{22}(x). \quad (4.14)$$

■

5 Second recursion formula for $Y(2|1)$ and $Y(1|2)$ BVs

To get the second recursion formula, one needs to use the morphism φ defined in section 2.3. To clarify the presentation, we use the following notation:

$$\varphi : \begin{cases} Y(2|1) \rightarrow Y(1|2) \\ T_{12}(u) \rightarrow -\tilde{T}_{23}(u) \\ T_{23}(u) \rightarrow \tilde{T}_{12}(u) \\ T_{13}(u) \rightarrow \tilde{T}_{13}(u) \\ \lambda_2(u) \rightarrow -\tilde{\lambda}_2(u) \end{cases} \quad \text{and} \quad \tilde{\varphi} : \begin{cases} Y(1|2) \rightarrow Y(2|1) \\ \tilde{T}_{12}(u) \rightarrow T_{23}(u) \\ \tilde{T}_{23}(u) \rightarrow -T_{12}(u) \\ \tilde{T}_{13}(u) \rightarrow T_{13}(u) \\ \tilde{\lambda}_2(u) \rightarrow -\lambda_2(u) \end{cases} \quad (5.1)$$

where we have explicitly written the image of the generators needed in the following.

Lemma 5.1. *The isomorphisms φ and $\tilde{\varphi}$ provide the following relations between $Y(1|2)$ BVs and $Y(2|1)$ ones:*

$$\varphi(\Phi_{ab}(\bar{u}, \bar{v})) = \tilde{\Phi}_{ba}(\bar{v}, \bar{u}) \quad \text{and} \quad \tilde{\varphi}(\tilde{\Phi}_{ab}(\bar{u}, \bar{v})) = \Phi_{ba}(\bar{v}, \bar{u}). \quad (5.2)$$

Proof: We start with an on-shell BV of $Y(1|2)$ and apply $\tilde{\varphi}$ to the equality (3.16), written in $Y(1|2)$. Since $\tilde{\varphi}$ is compatible with the supertrace, one has $\tilde{\varphi}(\tilde{t}(z)) = t(z)$. This shows that $\tilde{\varphi}(\tilde{\Phi}_{ab}(\bar{u}, \bar{v}))$ is an eigenvector of $t(z)$ in $Y(2|1)$. Acting with $\tilde{\varphi}$ on the relations (3.11), and using (5.1), it shows that it is proportional to $\Phi_{ba}(\bar{v}, \bar{u})$. In the same way, we show that $\varphi(\Phi_{ab}(\bar{u}, \bar{v}))$ is proportional to $\tilde{\Phi}_{ba}(\bar{v}, \bar{u})$.

To fix the normalisation in equality (5.2), we consider a specific coefficient in the super-trace formula. To get this coefficient, we will use

$$\text{In } Y(2|1) : \quad \text{str}(T(u) E_{21}) = T_{12}(u) \quad ; \quad \text{str}(T(u) E_{32}) = -T_{23}(u) \quad (5.3)$$

$$\text{In } Y(1|2) : \quad \text{str}(\tilde{T}(u) E_{21}) = -\tilde{T}_{12}(u) \quad ; \quad \text{str}(\tilde{T}(u) E_{32}) = -\tilde{T}_{23}(u). \quad (5.4)$$

Considering (3.2) written in $Y(2|1)$, the coefficient of $T_{12}(\bar{u}) T_{23}(\bar{v})$ in $\Phi_{ab}(\bar{u}, \bar{v})$ is $H(\bar{v})^{-1}$. Through the action of φ , it provides a term $(-1)^a \tilde{T}_{23}(\bar{u}) \tilde{T}_{12}(\bar{v})$. To get the corresponding term in $\tilde{\Phi}_{ab}(\bar{u}, \bar{v})$, we use the relation (2.8) to rewrite (4.13) as

$$\begin{aligned} \tilde{\Phi}_{a,b}(\bar{u}, \bar{v}) &= \frac{(-1)^a}{H(\bar{u})} \text{str}_{1 \dots a+b} \left[\mathbb{R}_{1 \dots a+b} \tilde{T}_{a+1}(v_1) \cdots \tilde{T}_{a+b}(v_b) \tilde{T}_1(u_1) \cdots \tilde{T}_a(u_a) \right. \\ &\quad \left. \times E_{21}^{(1)} \cdots E_{21}^{(a)} E_{32}^{(a+1)} \cdots E_{32}^{(a+b)} \right] \Omega. \end{aligned} \quad (5.5)$$

Then the coefficient of $\tilde{T}_{23}(\bar{u}) \tilde{T}_{12}(\bar{v})$ in $\tilde{\Phi}_{ba}(\bar{v}, \bar{u})$ is $(-1)^a H(\bar{v})^{-1}$, leading to the first relation in (5.2).

Finally, the coefficient of $\tilde{T}_{12}(\bar{u}) \tilde{T}_{23}(\bar{v})$ in $\tilde{\Phi}_{ab}(\bar{u}, \bar{v})$ is $(-1)^b H(\bar{u})^{-1}$, which, using $\tilde{\varphi}$, is sent to $H(\bar{u})^{-1} T_{23}(\bar{u}) T_{12}(\bar{v})$. The comparison with the coefficient of $T_{23}(\bar{u}) T_{12}(\bar{v})$ in $\Phi_{ba}(\bar{v}, \bar{u})$, which is $H(\bar{u})^{-1}$, obtained from (5.5), provides the second relation of (5.2). ■

Proposition 5.1. *The $Y(2|1)$ Bethe vectors obey the following recursion relation:*

$$\begin{aligned} \Phi_{ab}(\bar{u}, \bar{v}) &= h(\bar{v}_1, v_1)^{-1} T_{23}(v_1) \Phi_{a,b-1}(\bar{u}, \bar{v}_1) \\ &+ h(\bar{v}_1, v_1)^{-1} \sum_{j=1}^a \lambda_2(u_j) g(v_1, u_j) f(\bar{v}_1, u_j) f(\bar{u}_j, u_j) T_{13}(v_1) \Phi_{a-1,b-1}(\bar{u}_j, \bar{v}_1). \end{aligned} \quad (5.6)$$

The $Y(1|2)$ Bethe vectors obey the following recursion relation:

$$\begin{aligned} \tilde{\Phi}_{ab}(\bar{u}, \bar{v}) &= -\tilde{T}_{23}(v_1) \tilde{\Phi}_{a,b-1}(\bar{u}, \bar{v}_1) \\ &- \sum_{j=1}^a \tilde{\lambda}_2(u_j) g(u_j, v_1) f(u_j, \bar{v}_1) g(\bar{u}_j, u_j) \tilde{T}_{13}(v_1) \tilde{\Phi}_{a-1,b-1}(\bar{u}_j, \bar{v}_1). \end{aligned}$$

Proof: One starts with the recursion formula for $Y(1|2)$ BVs, as given in proposition 4.2 and rewritten as

$$\begin{aligned} \tilde{\Phi}_{ba}(\bar{v}, \bar{u}) &= h(\bar{v}_1, v_1)^{-1} \tilde{T}_{12}(v_1) \tilde{\Phi}_{b-1,a}(\bar{v}_1, \bar{u}) \\ &- h(\bar{v}_1, v_1)^{-1} \sum_{j=1}^a \tilde{\lambda}_2(u_j) g(v_1, u_j) f(\bar{v}_1, u_j) f(\bar{u}_j, u_j) \tilde{T}_{13}(v_1) \tilde{\Phi}_{a-1,b-1}(\bar{v}_1, \bar{u}_j). \end{aligned} \quad (5.7)$$

Now, applying the isomorphism φ , we get (5.6).

In the same way, starting from (4.1) written for $\Phi_{ba}(\bar{v}, \bar{u})$ and applying $\tilde{\varphi}$, we get (5.7). ■

Note that applying again the morphism φ (resp. $\tilde{\varphi}$) on relation (5.6) (resp. (5.7)), one gets back to the recursion relation (4.13) (resp. (4.1)).

6 Dual Bethe vectors

Using the antimorphism ψ , one can map BVs into dual BVs, that are left-eigenvectors of the transfer matrix (when on-shell). Again, to lighten the presentation, we focus on $Y(2|1)$ and $Y(1|2)$ BVs, but the technique applies also to the $Y(m|n)$ case. We define the dual Bethe vectors as

$$\Psi_{ab}(\bar{u}, \bar{v}) = \psi(\Phi_{ab}(\bar{u}^*, \bar{v}^*)). \quad (6.1)$$

Recall that for any set $\bar{w} = \{w_1, w_2, \dots, w_a\}$, we define its conjugate set as $\bar{w}^* = \{w_a, \dots, w_2, w_1\}$.

6.1 Supertrace formula

Proposition 6.1. *The $Y(2|1)$ dual BVs admit the following super-trace expression:*

$$\begin{aligned} \Psi_{ab}(\bar{u}, \bar{v}) &= \frac{(-1)^{b(b-1)/2}}{H(\bar{v}^*)} \Omega^\dagger \text{str}_{1\dots a+b} \left[T_1(u_1) \cdots T_a(u_a) T_{a+1}(v_1) \cdots T_{a+b}(v_b) \mathbb{R}_{1\dots a+b} \right. \\ &\quad \left. \times E_{12}^{(1)} \cdots E_{12}^{(a)} E_{23}^{(a+1)} \cdots E_{23}^{(a+b)} \right] \end{aligned} \quad (6.2)$$

where we noted $\psi(\Omega) = \Omega^\dagger$ and

$$H(\bar{v}^*) = \prod_{1 \leq j < k \leq b} h(v_j, v_k). \quad (6.3)$$

In the same way, $Y(1|2)$ dual BVs admit the following super-trace expression:

$$\begin{aligned} \tilde{\Psi}_{ab}(\bar{u}, \bar{v}) &= \frac{(-1)^{a(a-1)/2}}{H(\bar{u}^*)} \tilde{\Omega}^\dagger \text{str}_{1\dots a+b} \left[\tilde{T}_1(u_1) \cdots \tilde{T}_a(u_a) \tilde{T}_{a+1}(v_1) \cdots \tilde{T}_{a+b}(v_b) \mathbb{R}_{1\dots a+b} \right. \\ &\quad \left. \times E_{12}^{(1)} \cdots E_{12}^{(a)} E_{23}^{(a+1)} \cdots E_{23}^{(a+b)} \right] \end{aligned} \quad (6.4)$$

with $\tilde{\psi}(\tilde{\Omega}) = \tilde{\Omega}^\dagger$.

Proof: Applying ψ to the expression (3.2), one gets

$$\begin{aligned} \Psi_{ab}(\bar{u}^*, \bar{v}^*) &= \frac{(-1)^b}{H(\bar{v})} \Omega^\dagger \text{str}_{1\dots a+b} \left[\psi(T_{a+b}(v_b)) \cdots \psi(T_{a+1}(v_1)) \psi(T_a(u_a)) \cdots \psi(T_1(u_1)) \mathbb{R}_{1\dots a+b} \right. \\ &\quad \left. \times E_{21}^{(1)} \cdots E_{21}^{(a)} E_{32}^{(a+1)} \cdots E_{32}^{(a+b)} \right] \end{aligned} \quad (6.5)$$

$$\begin{aligned} &= \frac{(-1)^b}{H(\bar{v})} \Omega^\dagger \text{str}_{1\dots a+b} \left[T_{a+b}(v_b)^t \cdots T_{a+1}(v_1)^t T_a(u_a)^t \cdots T_1(u_1)^t \mathbb{R}_{1\dots a+b} \right. \\ &\quad \left. \times E_{21}^{(1)} \cdots E_{21}^{(a)} E_{32}^{(a+1)} \cdots E_{32}^{(a+b)} \right] \end{aligned} \quad (6.6)$$

$$\begin{aligned} &= \frac{(-1)^b}{H(\bar{v})} \Omega^\dagger \text{str}_{1\dots a+b} \left[\left(T_{a+b}(v_b) \cdots T_{a+1}(v_1) T_a(u_a) \cdots T_1(u_1) \right)^{t_1 \cdots t_{a+b}} \mathbb{R}_{1\dots a+b} \right. \\ &\quad \left. \times E_{21}^{(1)} \cdots E_{21}^{(a)} E_{32}^{(a+1)} \cdots E_{32}^{(a+b)} \right] \end{aligned} \quad (6.7)$$

$$\begin{aligned} &= \frac{(-1)^b}{H(\bar{v})} \Omega^\dagger \text{str}_{1\dots a+b} \left[T_{a+b}(v_b) \cdots T_{a+1}(v_1) T_a(u_a) \cdots T_1(u_1) \right. \\ &\quad \left. \times \left(\mathbb{R}_{1\dots a+b} E_{21}^{(1)} \cdots E_{21}^{(a)} E_{32}^{(a+1)} \cdots E_{32}^{(a+b)} \right)^{t_1^3 \cdots t_{a+b}^3} \right], \end{aligned} \quad (6.8)$$

where to get the last step we have used relation (2.5) for the cube of the transposition $(\cdot)^{t^3}$ and the fact that $(\cdot)^{t^4} = id$. Then, using $E_{21}^{t^3} = E_{12}$ and $E_{32}^{t^3} = -E_{23}$, we get

$$\begin{aligned} H(\bar{v}) \Psi_{ab}(\bar{u}^*, \bar{v}^*) &= \Omega^\dagger \text{str}_{1\dots a+b} \left[T_{a+b}(v_b) \cdots T_{a+1}(v_1) T_a(u_a) \cdots T_1(u_1) \right. \\ &\quad \left. \times E_{12}^{(1)} \cdots E_{12}^{(a)} E_{23}^{(a+1)} \cdots E_{23}^{(a+b)} \mathbb{R}_{1\dots a+b}^{t_1^3 \cdots t_{a+b}^3} \right]. \end{aligned} \quad (6.9)$$

Now, relabeling the spaces $(1, 2, \dots, a)$ to $(a, \dots, 2, 1)$ and $(a+1, a+2, \dots, a+b)$ to $(a+b, \dots, a+2, a+1)$, and the Bethe parameters accordingly, we get

$$H(\bar{v}^*) \Psi_{ab}(\bar{u}, \bar{v}) = \Omega^\dagger \text{str}_{1\dots a+b} \left[T_{a+1}(v_1) \cdots T_{a+b}(v_b) T_1(u_1) \cdots T_a(u_a) \right. \\ \left. \times E_{12}^{(a)} \cdots E_{12}^{(1)} E_{23}^{(a+b)} \cdots E_{23}^{(a+1)} \mathbb{R}_{a\dots 1, a+b\dots a+1}^{t_1^3 \cdots t_{a+b}^3} \right] \quad (6.10)$$

$$= \Omega^\dagger \text{str}_{1\dots a+b} \left[\mathbb{R}_{1\dots a+b} T_{a+1}(v_1) \cdots T_{a+b}(v_b) T_1(u_1) \cdots T_a(u_a) \right. \\ \left. \times E_{12}^{(a)} \cdots E_{12}^{(1)} E_{23}^{(a+b)} \cdots E_{23}^{(a+1)} \right], \quad (6.11)$$

where in the last step we have used the property $\mathbb{R}_{a\dots 1, a+b\dots a+1}^{t_1^3 \cdots t_{a+b}^3} = \mathbb{R}_{1\dots a+b}$ and cyclicity of the super-trace. Notice that $H(\bar{v})$ and $H(\bar{v}^*)$ are not equal: compare the definition (3.3) of $H(\bar{v})$ with (6.3).

Finally, using repetitively relation (2.8), one proves that

$$\mathbb{R}_{1\dots a+b} T_{a+1}(v_1) \cdots T_{a+b}(v_b) T_1(u_1) \cdots T_a(u_a) = T_1(u_1) \cdots T_a(u_a) T_{a+1}(v_1) \cdots T_{a+b}(v_b) \mathbb{R}_{1\dots a+b}$$

and one gets (6.2) after reordering the tensor product $E_{12}^{(a)} \cdots E_{12}^{(1)} E_{23}^{(a+b)} \cdots E_{23}^{(a+1)}$. The same calculation starting from (4.12), with now $E_{21}^3 = -E_{12}$ and $E_{32}^3 = E_{23}$, leads to (6.4). ■

6.2 Recursion formulas

Proposition 6.2. *The $Y(2|1)$ dual BVs obey the following recursion relations*

$$\Psi_{ab}(\bar{u}, \bar{v}) = \Psi_{a-1,b}(\bar{u}_a, \bar{v}) T_{21}(u_a) \\ + (-1)^{b-1} \sum_{j=1}^b \lambda_2(v_j) g(v_j, u_a) f(v_j, \bar{u}_a) g(\bar{v}_j, v_j) \Psi_{a-1,b-1}(\bar{u}_a, \bar{v}_j) T_{31}(u_a), \quad (6.12)$$

$$\Psi_{ab}(\bar{u}, \bar{v}) = (-1)^{b-1} h(\bar{v}_b, v_b)^{-1} \Psi_{a,b-1}(\bar{u}, \bar{v}_b) T_{32}(v_b) \\ + (-1)^{b-1} h(\bar{v}_b, v_b)^{-1} \sum_{j=1}^a \lambda_2(u_j) g(v_b, u_j) f(\bar{v}_b, u_j) f(\bar{u}_j, u_j) \Psi_{a-1,b-1}(\bar{u}_j, \bar{v}_b) T_{31}(v_b). \quad (6.13)$$

The $Y(1|2)$ dual BVs obey the following recursion relations

$$\tilde{\Psi}_{ab}(\bar{u}, \bar{v}) = (-1)^{a-1} h(\bar{u}_a, u_a)^{-1} \tilde{\Psi}_{a-1,b}(\bar{u}_a, \bar{v}) \tilde{T}_{21}(u_a) \\ - (-1)^{a-1} h(\bar{u}_a, u_a)^{-1} \sum_{j=1}^b \tilde{\lambda}_2(v_j) g(u_a, v_j) f(\bar{u}_a, v_j) f(\bar{v}_j, v_j) \tilde{\Psi}_{a-1,b-1}(\bar{u}_a, \bar{v}_j) \tilde{T}_{31}(u_a), \quad (6.14)$$

$$\tilde{\Psi}_{ab}(\bar{u}, \bar{v}) = -\tilde{\Psi}_{a,b-1}(\bar{u}, \bar{v}_b) \tilde{T}_{32}(v_b) \\ - (-1)^{a-1} \sum_{j=1}^a \tilde{\lambda}_2(u_j) g(u_j, v_b) f(u_j, \bar{v}_b) g(\bar{u}_j, u_j) \tilde{\Psi}_{a-1,b-1}(\bar{u}_j, \bar{v}_b) \tilde{T}_{31}(v_b). \quad (6.15)$$

Proof: Applying ψ to the relation (4.1), we get

$$\begin{aligned}\Psi_{ab}(\bar{u}^*, \bar{v}^*) &= \Psi_{a-1,b}(\bar{u}_1^*, \bar{v}^*) T_{21}(u_1) \\ &+ \sum_{j=1}^b (-1)^{b-1} \lambda_2(v_j) g(v_j, u_1) f(v_j, \bar{u}_1) g(\bar{v}_j, v_j) \Psi_{a-1,b-1}(\bar{u}_1^*, \bar{v}_j^*) T_{31}(u_1)\end{aligned}\quad (6.16)$$

where \bar{u}_1^* is the conjugate of the set \bar{u}_1 and we remind that ψ acts as

$$\psi(T_{12}(u)) = T_{21}(u), \quad \psi(T_{23}(u)) = T_{32}(u), \quad \psi(T_{13}(u)) = T_{31}(u).$$

Now, we set $k = b + 1 - j$, to get

$$\begin{aligned}\Psi_{ab}(\bar{u}^*, \bar{v}^*) &= \Psi_{a-1,b}(\bar{u}_1^*, \bar{v}^*) T_{21}(u_1) + \sum_{k=1}^b (-1)^{b-1} \lambda_2(v_{b+1-k}) g(v_{b+1-k}, u_1) f(v_{b+1-k}, \bar{u}_1) \\ &\times g(\bar{v}_{b+1-k}, v_{b+1-k}) \Psi_{a-1,b-1}(\bar{u}_1^*, \bar{v}_{b+1-k}^*) T_{31}(u_1).\end{aligned}\quad (6.17)$$

Finally, to get relation (6.12), one relabels the Bethe parameters as $v'_j = v_{b+1-j}$ and $u'_j = u_{a+1-j}$.

The proof for relation (6.13) follows the same lines. Relations (6.14) and (6.15) are proven the same way, taking into account the different \mathbb{Z}_2 -grading. \blacksquare

6.3 Morphisms on dual vectors

To be complete, we also provide relations between BVs and dual BVs. As usual, Φ and Ψ refer to $Y(2|1)$ while $\tilde{\Phi}$ and $\tilde{\Psi}$ correspond to $Y(1|2)$.

Lemma 6.1. *We have the following relations*

$$\psi(\Psi_{ab}(\bar{u}, \bar{v})) = \Phi_{ab}(\bar{u}^*, \bar{v}^*), \quad \tilde{\psi}(\tilde{\Psi}_{ab}(\bar{u}, \bar{v})) = \tilde{\Phi}_{ab}(\bar{u}^*, \bar{v}^*) \quad (6.18)$$

$$\varphi(\Psi_{ab}(\bar{u}, \bar{v})) = \tilde{\Psi}_{ba}(\bar{v}, \bar{u}), \quad \tilde{\varphi}(\tilde{\Psi}_{ab}(\bar{u}, \bar{v})) = \Psi_{ba}(\bar{v}, \bar{u}). \quad (6.19)$$

Proof: Relations (6.18) and (6.19) are a direct consequence of lemma 2.1. For instance

$$\varphi(\Psi_{ab}(\bar{u}, \bar{v})) = \varphi \circ \psi(\Phi_{ab}(\bar{u}^*, \bar{v}^*)) = \tilde{\psi} \circ \varphi(\Phi_{ab}(\bar{u}^*, \bar{v}^*)) = \tilde{\psi}(\tilde{\Phi}_{ba}(\bar{v}^*, \bar{u}^*)) = \tilde{\Psi}_{ba}(\bar{v}, \bar{u}). \quad (6.20)$$

The remaining relations are proven the same way. \blacksquare

7 Explicit expressions for $Y(2|1)$ Bethe vectors

From the recursion formulas, one can deduce explicit expressions for BVs and dual BVs. This section is devoted to the proof of the following proposition:

Proposition 7.1. *In $Y(2|1)$ the Bethe vectors admit the two following explicit expressions*

$$\Phi_{a,b}(\bar{u}, \bar{v}) = \sum g(\bar{v}_I, \bar{u}_I) f(\bar{u}_I, \bar{u}_{II}) g(\bar{v}_{II}, \bar{v}_I) h(\bar{u}_I, \bar{u}_I) \mathbb{T}_{13}(\bar{u}_I) T_{12}(\bar{u}_{II}) \mathbb{T}_{23}(\bar{v}_{II}) \lambda_2(\bar{v}_I) \Omega, \quad (7.1)$$

$$\Phi_{a,b}(\bar{u}, \bar{v}) = \sum K_\ell(\bar{v}_I | \bar{u}_I) f(\bar{u}_I, \bar{u}_{II}) g(\bar{v}_{II}, \bar{v}_I) \mathbb{T}_{13}(\bar{v}_I) \mathbb{T}_{23}(\bar{v}_{II}) T_{12}(\bar{u}_{II}) \lambda_2(\bar{u}_I) \Omega. \quad (7.2)$$

where we have introduced the notation (for $\#\bar{u} = \#\bar{v} = \ell$):

$$\mathbb{T}_{23}(\bar{v}) = \frac{1}{H(\bar{v})} \prod_{1 \leq j \leq \ell}^{\rightarrow} T_{23}(v_j) \quad \text{and} \quad \mathbb{T}_{13}(\bar{v}) = \frac{1}{H(\bar{v})} \prod_{1 \leq j \leq \ell}^{\rightarrow} T_{13}(v_j), \quad (7.3)$$

$$K_\ell(\bar{v} | \bar{u}) = \Delta_\ell(\bar{v}) \Delta'_\ell(\bar{u}) h(\bar{v}, \bar{u}) \det_\ell \left(\frac{g(v_j, u_k)}{h(v_j, u_k)} \right) \quad (7.4)$$

$$\Delta_\ell(\bar{v}) = \prod_{\ell \geq j > k \geq 1} g(v_j, v_k), \quad \Delta'_\ell(\bar{u}) = \prod_{\ell \geq j > k \geq 1} g(u_k, u_j). \quad (7.5)$$

In relations (7.1) and (7.2), the sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_{II}\}$ and $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$ with the restriction $\#\bar{u}_I = \#\bar{v}_I = \ell$, $\ell = 0, 1, \dots, \min(a, b)$.

Actually, we will prove a property which is slightly more general. Define

$$X_{a,b}(\bar{u}, \bar{v}) = \sum g(\bar{v}_I, \bar{u}_I) f(\bar{u}_I, \bar{u}_{II}) g(\bar{v}_{II}, \bar{v}_I) h(\bar{u}_I, \bar{u}_I) \mathbb{T}_{13}(\bar{u}_I) T_{12}(\bar{u}_{II}) \mathbb{T}_{23}(\bar{v}_{II}) T_{22}(\bar{v}_I). \quad (7.6)$$

The indices a, b in $X_{a,b}(\bar{u}, \bar{v})$ indicate that $\#\bar{u} = a$ and $\#\bar{v} = b$ and the sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_{II}\}$ and $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$ with the restriction $\#\bar{u}_I = \#\bar{v}_I$.

We are going to prove that the operator $X_{a,b}(\bar{u}, \bar{v})$ satisfies a recursion

$$X_{a,b}(\bar{u}, \bar{v}) = T_{12}(u_a) X_{a-1,b}(\bar{u}_a, \bar{v}) + \sum_{j=1}^b g(v_j, u_a) f(v_j, \bar{u}_a) g(\bar{v}_j, v_j) T_{13}(u_a) X_{a-1,b-1}(\bar{u}_a, \bar{v}_j) T_{22}(v_j). \quad (7.7)$$

Applying this relation on the highest weight vector Ω then shows that $X_{a,b}(\bar{u}, \bar{v})\Omega$ obeys the first recursion relation for $\Phi_{ab}(\bar{u}, \bar{v})$. Since they coincide for $a = 0$, it proves that they are equal, so that (7.6) provides the explicit expression (7.1) for $\Phi_{ab}(\bar{u}, \bar{v})$.

In the same way, starting from

$$Y_{a,b}(\bar{u}, \bar{v}) = \sum K_\ell(\bar{v}_I | \bar{u}_I) f(\bar{u}_I, \bar{u}_{II}) g(\bar{v}_{II}, \bar{v}_I) \mathbb{T}_{13}(\bar{v}_I) \mathbb{T}_{23}(\bar{v}_{II}) T_{12}(\bar{u}_{II}) T_{22}(\bar{u}_I), \quad (7.8)$$

where the sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_{II}\}$ and $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$ with the restriction $\#\bar{u}_I = \#\bar{v}_I = \ell$, where $\ell = 0, 1, \dots, \min(a, b)$, we will show that the operator $Y_{a,b}(\bar{u}, \bar{v})$ satisfies a recursion

$$h(\bar{v}, v_b) Y_{a,b}(\bar{u}, \bar{v}) = T_{23}(v_b) Y_{a,b-1}(\bar{u}, \bar{v}_b) + \sum_{j=1}^a g(v_b, u_j) f(\bar{v}_b, u_j) f(u_j, \bar{u}_j) T_{13}(v_b) Y_{a-1,b-1}(\bar{u}_j, \bar{v}_b) T_{22}(u_j). \quad (7.9)$$

Again, application of this relation on Ω will show that $Y_{a,b}(\bar{u}, \bar{v})\Omega$ and $\Phi_{a,b}(\bar{u}, \bar{v})$ coincide, and we get the second explicit expression (7.2) for $\Phi_{a,b}(\bar{u}, \bar{v})$.

7.1 Multiple commutation relations

Before proving the recursion relations for $X_{a,b}(\bar{u}, \bar{v})$ and $Y_{a,b}(\bar{u}, \bar{v})$, we need some formulas for multiple commutation relations.

Lemma 7.1. *Let $\#\bar{u} = \ell$. Then, for $\mathbb{T}_{13}(\bar{u})$ defined as in (7.3), we have*

$$T_{12}(v)\mathbb{T}_{13}(\bar{u}) = f(\bar{u}, v)\mathbb{T}_{13}(\bar{u})T_{12}(v) + \sum_{k=1}^{\ell} g(v, u_k)g(\bar{u}_k, u_k)T_{13}(v)\mathbb{T}_{13}(\bar{u}_k)T_{12}(u_k), \quad (7.10)$$

$$T_{23}(v)\mathbb{T}_{13}(\bar{u}) = (-1)^{\ell}f(\bar{u}, v)\mathbb{T}_{13}(\bar{u})T_{23}(v) + \sum_{k=1}^{\ell} g(u_k, v)g(u_k, \bar{u}_k)T_{13}(v)\mathbb{T}_{13}(\bar{u}_k)T_{23}(u_k). \quad (7.11)$$

Proof: We have from the RTT -relation:

$$T_{12}(v)T_{13}(u) = f(u, v)T_{13}(u)T_{12}(v) + g(v, u)T_{13}(v)T_{12}(u). \quad (7.12)$$

Then we consider the case of one operator $T_{12}(v)$ and ℓ operators $T_{13}(u_k)$. We use the standard approach of the algebraic Bethe ansatz. It is clear that

$$T_{12}(v)\mathbb{T}_{13}(\bar{u}) = \Lambda\mathbb{T}_{13}(\bar{u})T_{12}(v) + \sum_{k=1}^{\ell} \Lambda_k T_{13}(v)\mathbb{T}_{13}(\bar{u}_k)T_{12}(u_k), \quad (7.13)$$

where Λ and Λ_k are some rational coefficients. In order to find Λ one should ignore the second term in the r.h.s. of (7.12). Then

$$\Lambda = f(\bar{u}, v). \quad (7.14)$$

Now let us find Λ_k . Due to the symmetry of $\mathbb{T}_{13}(\bar{u})$ over \bar{u} it is enough to find Λ_1 only. We have

$$T_{12}(v)\mathbb{T}_{13}(\bar{u}) = T_{12}(v) \frac{T_{13}(u_1) \dots T_{13}(u_{\ell})}{\prod_{\ell \geq j > k \geq 1} h(u_j, u_k)} \quad (7.15)$$

$$= g(v, u_1) \frac{T_{13}(v)T_{12}(u_1)T_{13}(u_2) \dots T_{13}(u_{\ell})}{\prod_{\ell \geq j > k \geq 1} h(u_j, u_k)} + UWT, \quad (7.16)$$

where UWT means *unwanted terms*. Then the operator $T_{12}(u_1)$ should move to the right keeping its argument, what gives us

$$T_{12}(v)\mathbb{T}_{13}(\bar{u}) = g(v, u_1)f(\bar{u}_1, u_1) \frac{T_{13}(v)T_{13}(u_2) \dots T_{13}(u_{\ell})}{\prod_{\ell \geq j > k \geq 1} h(u_j, u_k)} T_{12}(u_1) + UWT \quad (7.17)$$

$$= g(v, u_1)g(\bar{u}_1, u_1)T_{13}(v)\mathbb{T}_{13}(\bar{u}_1)T_{12}(u_1) + UWT. \quad (7.18)$$

Thus, we obtain

$$\Lambda_1 = g(v, u_1)g(\bar{u}_1, u_1), \quad (7.19)$$

and, hence,

$$\Lambda_k = g(v, u_k)g(\bar{u}_k, u_k). \quad (7.20)$$

Thus, we get (7.10).

In the same way, starting from the *RTT*-relation

$$T_{23}(v)T_{13}(u) = -f(u, v)T_{13}(u)T_{23}(v) - g(v, u)T_{13}(v)T_{23}(u), \quad (7.21)$$

similar considerations show relation (7.11). ■

7.2 Proof of the recursion for $X_{a,b}(\bar{u}, \bar{v})$

Here we prove that $X_{a,b}(\bar{u}, \bar{v})$ defined by (7.6) satisfies the recursion (7.7).

Acting with $T_{12}(u_a)$ onto $X_{a-1,b}(\bar{u}_a; \bar{v})$ we should move $T_{12}(u_a)$ through the product $\mathbb{T}_{13}(\bar{u}_1)$. For this we can use (7.13). It is convenient to rewrite it in the following form:

$$T_{12}(v)\mathbb{T}_{13}(\bar{u}) = f(\bar{u}, v)\mathbb{T}_{13}(\bar{u})T_{12}(v) + \sum g(v, u_i)g(\bar{u}_{ii}, u_i)T_{13}(v)\mathbb{T}_{13}(\bar{u}_{ii})T_{12}(u_i), \quad (7.22)$$

where the sum is taken over partitions $\bar{u} \Rightarrow \{u_i, \bar{u}_{ii}\}$ with $\#u_i = 1$ (therefore we do not write bar for this subset). Let us call the first term in the r.h.s. of (7.22) *direct action*, while the remaining sum over partitions is called *indirect action*.

We have

$$T_{12}(u_a)X_{a-1,b}(\bar{u}_a; \bar{v}) = M_1 + T_{13}(u_a)M_2, \quad (7.23)$$

where M_1 and M_2 respectively correspond to the direct and indirect actions of $T_{12}(u_a)$. Consider first the contribution M_1 . We have

$$M_1 = \sum g(\bar{v}_1, \bar{u}_1)f(\bar{u}_1, \bar{u}_{II})g(\bar{v}_{II}, \bar{v}_1)h(\bar{u}_1, \bar{u}_1)f(\bar{u}_1, u_a) \mathbb{T}_{13}(\bar{u}_1) T_{12}(\bar{u}_{II})T_{12}(u_a) \mathbb{T}_{23}(\bar{v}_{II})T_{22}(\bar{v}_1). \quad (7.24)$$

Here the sum is taken over partitions $\bar{u}_a \Rightarrow \{\bar{u}_1, \bar{u}_{II}\}$ and $\bar{v} \Rightarrow \{\bar{v}_1, \bar{v}_{II}\}$ with $\#\bar{v}_1 = \#\bar{u}_1$. Recall also that $\bar{u}_a = \bar{u} \setminus \{u_a\}$. Denoting $\{\bar{u}_{II}, u_a\} = \bar{u}_{II'}$ we obtain

$$M_1 = \sum_{u_a \in \bar{u}_{II'}} g(\bar{v}_1, \bar{u}_1)f(\bar{u}_1, \bar{u}_{II'})g(\bar{v}_{II}, \bar{v}_1)h(\bar{u}_1, \bar{u}_1) \mathbb{T}_{13}(\bar{u}_1) T_{12}(\bar{u}_{II'}) \mathbb{T}_{23}(\bar{v}_{II})T_{22}(\bar{v}_1), \quad (7.25)$$

where now the sum is taken over partitions $\bar{u} \Rightarrow \{\bar{u}_1, \bar{u}_{II'}\}$ and $\bar{v} \Rightarrow \{\bar{v}_1, \bar{v}_{II}\}$ with $\#\bar{v}_1 = \#\bar{u}_1$ and an additional restriction $u_a \in \bar{u}_{II'}$. Clearly, if we ignore the latest restriction, then we obtain $X_{a,b}(\bar{u}, \bar{v})$. Therefore

$$M_1 = X_{a,b}(\bar{u}, \bar{v}) - \sum_{u_a \in \bar{u}_I} g(\bar{v}_1, \bar{u}_1)f(\bar{u}_1, \bar{u}_{II'})g(\bar{v}_{II}, \bar{v}_1)h(\bar{u}_1, \bar{u}_1) \mathbb{T}_{13}(\bar{u}_1) T_{12}(\bar{u}_{II'}) \mathbb{T}_{23}(\bar{v}_{II'})T_{22}(\bar{v}_1), \quad (7.26)$$

where now the restriction is $u_a \in \bar{u}_I$. Setting $\bar{u}_I = \{u_a, \bar{u}_{II}\}$, we obtain

$$M_1 - X_{a,b}(\bar{u}, \bar{v}) = - \sum g(\bar{v}_1, \bar{u}_{II})g(\bar{v}_1, u_a)f(u_a, \bar{u}_{II'})f(\bar{u}_{II}, \bar{u}_{II'})g(\bar{v}_{II}, \bar{v}_1) \\ \times h(u_a, \bar{u}_{II})h(\bar{u}_{II}, \bar{u}_{II'})T_{13}(u_a)\mathbb{T}_{13}(\bar{u}_{II}) T_{12}(\bar{u}_{II'}) \mathbb{T}_{23}(\bar{v}_{II})T_{22}(\bar{v}_1), \quad (7.27)$$

where we have extracted explicitly $T_{13}(u_a)$ from the product $\mathbb{T}_{13}(\bar{u}_I)$. Then we recast (7.27) as follows

$$M_1 - X_{a,b}(\bar{u}, \bar{v}) = -T_{13}(u_a) f(u_a, \bar{u}_a) \sum g(\bar{v}_I, \bar{u}_{II}) \frac{g(\bar{v}_I, u_a)}{g(u_a, \bar{u}_{II})} f(\bar{u}_{II}, \bar{u}_{II'}) g(\bar{v}_{II}, \bar{v}_I) h(\bar{u}_{II}, \bar{u}_{II}) \\ \times \mathbb{T}_{13}(\bar{u}_{II}) T_{12}(\bar{u}_{II'}) \mathbb{T}_{23}(\bar{v}_{II}) T_{22}(\bar{v}_I). \quad (7.28)$$

The sum is taken over partitions $\bar{u}_a \Rightarrow \{\bar{u}_{II}, \bar{u}_{II'}\}$ and $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_{II}\}$ with $\#\bar{v}_I = \#\bar{u}_{II} + 1$.

The final step of the transformations of M_1 is to develop the ratio $g(\bar{v}_I, u_a)/g(u_a, \bar{u}_{II})$ over the poles at $u_a = v_i \in \bar{v}_I$:

$$\frac{g(\bar{v}_I, u_a)}{g(u_a, \bar{u}_{II})} = \sum g(v_i, u_a) \frac{g(\bar{v}_{II}, v_i)}{g(v_i, \bar{u}_{II})}. \quad (7.29)$$

Here the sum is taken over partitions $\bar{v}_I \Rightarrow \{v_i, \bar{v}_{II}\}$, where v_i consists of one element. Substituting this into (7.27) and setting there $\bar{v}_I = \{v_i, \bar{v}_{II}\}$ we obtain

$$M_1 - X_{a,b}(\bar{u}, \bar{v}) = -T_{13}(u_a) f(u_a, \bar{u}_a) \sum g(v_i, u_a) g(\bar{v}_{II}, \bar{u}_{II}) f(\bar{u}_{II}, \bar{u}_{II'}) h(\bar{u}_{II}, \bar{u}_{II}) \\ \times g(\bar{v}_{II}, \bar{v}_{II}) g(\bar{v}_{II}, v_i) g(\bar{v}_{II}, v_i) \mathbb{T}_{13}(\bar{u}_{II}) T_{12}(\bar{u}_{II'}) \mathbb{T}_{23}(\bar{v}_{II}) T_{22}(\bar{v}_{II}) T_{22}(v_i). \quad (7.30)$$

This is the final expression for the contribution M_1 . The sum is organized as follows. The set \bar{u}_a is divided into two subsets $\bar{u}_a \Rightarrow \{\bar{u}_{II}, \bar{u}_{II'}\}$. The set \bar{v} is divided into three subsets $\bar{v} \Rightarrow \{v_i, \bar{v}_{II}, \bar{v}_{II'}\}$. The restrictions are $\#\bar{v}_{II} = \#\bar{u}_{II}$ and $\#v_i = 1$.

Consider now the result of the indirect action M_2 . Using (7.22) we obtain

$$M_2 = \sum g(\bar{v}_I, \bar{u}_I) f(\bar{u}_I, \bar{u}_{II}) g(\bar{v}_{II}, \bar{v}_I) h(\bar{u}_I, \bar{u}_I) g(u_a, u_i) g(\bar{u}_{II}, u_i) \\ \times \mathbb{T}_{13}(\bar{u}_{II}) T_{12}(\bar{u}_{II}) T_{12}(u_i) \mathbb{T}_{23}(\bar{v}_{II}) T_{22}(\bar{v}_I). \quad (7.31)$$

Here the set \bar{u}_a is divided into $\bar{u}_a \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$, and then the subset \bar{u}_I is divided once more $\bar{u}_I \Rightarrow \{u_i, \bar{u}_{II}\}$, where u_i consists of only one element.

Let $\{\bar{u}_I, \bar{u}_{II}\} = \bar{u}_{II'}$. Then we can recast (7.31) as follows:

$$M_2 = \sum \frac{g(\bar{v}_I, u_i)}{g(u_i, \bar{u}_{II})} g(\bar{v}_I, \bar{u}_{II}) f(u_i, \bar{u}_0) g(u_a, u_i) f(\bar{u}_{II}, \bar{u}_{II'}) g(\bar{v}_{II}, \bar{v}_I) h(\bar{u}_{II}, \bar{u}_{II}) \\ \times \mathbb{T}_{13}(\bar{u}_{II}) T_{12}(\bar{u}_{II'}) \mathbb{T}_{23}(\bar{v}_{II}) T_{22}(\bar{v}_I), \quad (7.32)$$

where we also substituted $\bar{u}_I = \{u_i, \bar{u}_{II}\}$ and set $\bar{u}_0 = \bar{u}_a \setminus \{u_i\}$. Now we again develop the ratio $g(\bar{v}_I, u_i)/g(u_i, \bar{u}_{II})$ over the poles:

$$\frac{g(\bar{v}_I, u_i)}{g(u_i, \bar{u}_{II})} = \sum g(v_i, u_i) \frac{g(\bar{v}_{II}, v_i)}{g(v_i, \bar{u}_{II})}, \quad (7.33)$$

where the sum is taken over partitions $\bar{v}_I \Rightarrow \{v_i, \bar{v}_{II}\}$, and v_i consists of one element only. Then (7.32) takes the form

$$M_2 = \sum \left[g(u_a, u_i) g(v_i, u_i) f(u_i, \bar{u}_0) \right] g(\bar{v}_{II}, \bar{u}_{II}) f(\bar{u}_{II}, \bar{u}_{II'}) h(\bar{u}_{II}, \bar{u}_{II}) \\ \times g(\bar{v}_{II}, v_i) g(\bar{v}_{II}, v_i) g(\bar{v}_{II}, \bar{v}_{II}) \mathbb{T}_{13}(\bar{u}_{II}) T_{12}(\bar{u}_{II'}) \mathbb{T}_{23}(\bar{v}_{II}) T_{22}(\bar{v}_{II}) T_{22}(v_i). \quad (7.34)$$

Here the sum is taken over partitions $\bar{u}_a \Rightarrow \{u_i, \bar{u}_{ii}, \bar{u}_{i' i'}\}$ and $\bar{v} \Rightarrow \{v_i, \bar{v}_{ii}, \bar{v}_{i' i'}\}$ with the restrictions $\#\bar{v}_{ii} = \#\bar{u}_{ii}$ and $\#v_i = \#u_i = 1$. The sum over u_i (see the terms in the square brackets in (7.34)) can be computed via a special contour integral. Let

$$\sum_{u_i \in \bar{u}_a} f(u_i, \bar{u}_0) g(u_a, u_i) g(v_i, u_i) = J. \quad (7.35)$$

Consider an integral

$$I = \frac{c}{2\pi i} \oint_{|z|=R \rightarrow \infty} \frac{dz}{(u_a - z)(v_i - z)} \prod_{k=1}^{a-1} \frac{z - u_k + c}{z - u_k}, \quad (7.36)$$

where the integration contour is $|z| = R \rightarrow \infty$. Obviously, $I = 0$, because the integrand behaves as z^{-2} at $|z| \rightarrow \infty$. On the other hand, this integral is equal to the sum of residues within the integration contour. The sum of the residues in the poles $z = u_k$ gives J . Two additional contributions come from the poles at $z = u_a$ and $z = v_i$. Thus, we obtain

$$0 = J - g(v_i, u_a) f(u_a, \bar{u}_a) - g(v_i, u_a) f(v_i, \bar{u}_a). \quad (7.37)$$

Substituting this into (7.34) we find

$$\begin{aligned} M_2 = \sum g(v_i, u_a) \Big\{ & f(u_a, \bar{u}_a) - f(v_i, \bar{u}_a) \Big\} g(\bar{v}_{ii}, \bar{u}_{ii}) f(\bar{u}_{ii}, \bar{u}_{i' i'}) h(\bar{u}_{ii}, \bar{u}_{ii}) \\ & \times g(\bar{v}_{ii}, v_i) g(\bar{v}_{i' i'}, v_i) g(\bar{v}_{i' i'}, \bar{v}_{ii}) \mathbb{T}_{13}(\bar{u}_{ii}) T_{12}(\bar{u}_{i' i'}) \mathbb{T}_{23}(\bar{v}_{i' i'}) T_{22}(\bar{v}_{ii}) T_{22}(v_i). \end{aligned} \quad (7.38)$$

We see that the first term in the braces cancels the contribution (7.30). Thus, we arrive at

$$\begin{aligned} T_{12}(u_a) X_{a-1, b}(\bar{u}_a, \bar{v}) - X_{a, b}(\bar{u}, \bar{v}) = & -T_{13}(u_a) \sum g(\bar{v}_{ii}, \bar{u}_{ii}) g(v_i, u_a) f(v_i, \bar{u}_a) f(\bar{u}_{ii}, \bar{u}_{i' i'}) h(\bar{u}_{ii}, \bar{u}_{ii}) \\ & \times g(\bar{v}_{ii}, v_i) g(\bar{v}_{i' i'}, v_i) g(\bar{v}_{i' i'}, \bar{v}_{ii}) \mathbb{T}_{13}(\bar{u}_{ii}) T_{12}(\bar{u}_{i' i'}) \mathbb{T}_{23}(\bar{v}_{i' i'}) T_{22}(\bar{v}_{ii}) T_{22}(v_i). \end{aligned} \quad (7.39)$$

Here the sum is taken over partitions $\bar{u}_a \Rightarrow \{\bar{u}_{ii}, \bar{u}_{i' i'}\}$ and $\bar{v} \Rightarrow \{v_i, \bar{v}_{ii}, \bar{v}_{i' i'}\}$ with the restrictions $\#\bar{v}_{ii} = \#\bar{u}_{ii}$ and $\#v_i = 1$. Let $\bar{v}_0 = \bar{v} \setminus \{v_i\} = \{\bar{v}_{ii}, \bar{v}_{i' i'}\}$. Then (7.39) takes the form

$$\begin{aligned} T_{12}(u_a) X_{a-1, b}(\bar{u}_a, \bar{v}) - X_{a, b}(\bar{u}, \bar{v}) = & -T_{13}(u_a) \sum g(v_i, u_a) f(v_i, \bar{u}_a) g(\bar{v}_0, v_i) \\ & \times \Big\{ g(\bar{v}_{ii}, \bar{u}_{ii}) f(\bar{u}_{ii}, \bar{u}_{i' i'}) g(\bar{v}_{i' i'}, \bar{v}_{ii}) h(\bar{u}_{ii}, \bar{u}_{ii}) \mathbb{T}_{13}(\bar{u}_{ii}) T_{12}(\bar{u}_{i' i'}) \mathbb{T}_{23}(\bar{v}_{i' i'}) T_{22}(\bar{v}_{ii}) \Big\} T_{22}(v_i). \end{aligned} \quad (7.40)$$

The sum over partitions in the braces evidently gives $X_{a-1, b-1}(\bar{u}_a, \bar{v}_i)$, and we finally obtain

$$T_{12}(u_a) X_{a-1, b}(\bar{u}_a, \bar{v}) = X_{a, b}(\bar{u}, \bar{v}) - \sum g(v_i, u_a) f(v_i, \bar{u}_a) g(\bar{v}_0, v_i) T_{13}(u_a) X_{a-1, b-1}(\bar{u}_a, \bar{v}_i) T_{22}(v_i). \quad (7.41)$$

This is exactly the recursion that we need.

7.3 Proof of the recursion for $Y_{a,b}(\bar{u}, \bar{v})$

The proof is very similar to the one given in section 7.2. Now we should multiply the operator $Y_{a,b-1}(\bar{u}, \bar{v}_b)$ by $T_{23}(v_b)$ from the left and move $T_{23}(v_b)$ through the product $\mathbb{T}_{13}(\bar{v}_1)$. The result can be written as a sum of two terms:

$$T_{23}(v_b)Y_{a,b-1}(\bar{u}, \bar{v}_b) = M_1 + T_{13}(v_b)M_2. \quad (7.42)$$

Here the contributions M_1 and M_2 respectively correspond to the first and the second terms in the action (7.11). Consider the first contribution

$$M_1 = \sum (-1)^\ell K_\ell(\bar{v}_1|\bar{u}_1)f(\bar{u}_1, \bar{u}_\Pi)g(\bar{v}_\Pi, \bar{v}_1)f(\bar{v}_1, v_b) \mathbb{T}_{13}(\bar{v}_1) T_{23}(v_b)\mathbb{T}_{23}(\bar{v}_\Pi) T_{12}(\bar{u}_\Pi)T_{22}(\bar{u}_1). \quad (7.43)$$

Here the sum is taken over partitions $\bar{v}_b \Rightarrow \{\bar{v}_1, \bar{v}_\Pi\}$ and $\bar{u} \Rightarrow \{\bar{u}_1, \bar{u}_\Pi\}$ with the restriction $\#\bar{u}_1 = \#\bar{v}_1 = \ell$. Combining $\{v_b, \bar{v}_\Pi\} = \bar{v}_{\Pi'}$, we obtain

$$M_1 = h(\bar{v}_b, v_b) \sum_{v_b \in \bar{v}_{\Pi'}} K_\ell(\bar{v}_1|\bar{u}_1)f(\bar{u}_1, \bar{u}_\Pi)g(\bar{v}_{\Pi'}, \bar{v}_1) \mathbb{T}_{13}(\bar{v}_1) \mathbb{T}_{23}(\bar{v}_{\Pi'}) T_{12}(\bar{u}_\Pi)T_{22}(\bar{u}_1), \quad (7.44)$$

where now the sum is taken over partitions of the complete set \bar{v} into subsets \bar{v}_1 and $\bar{v}_{\Pi'}$. However, we have the restriction $v_b \in \bar{v}_{\Pi'}$. Obviously, if we get rid of this restriction, then we obtain $h(\bar{v}_b, v_b)Y_{a,b}(\bar{u}, \bar{v})$. Thus,

$$\begin{aligned} M_1 - h(\bar{v}_b, v_b)Y_{a,b}(\bar{u}, \bar{v}) \\ = -h(\bar{v}_b, v_b) \sum_{v_b \in \bar{v}_1} K_\ell(\bar{v}_1|\bar{u}_1)f(\bar{u}_1, \bar{u}_\Pi)g(\bar{v}_{\Pi'}, \bar{v}_1) \mathbb{T}_{13}(\bar{v}_1) \mathbb{T}_{23}(\bar{v}_{\Pi'}) T_{12}(\bar{u}_\Pi)T_{22}(\bar{u}_1). \end{aligned} \quad (7.45)$$

Here the sum over partitions is the same as in (7.44) except that now the restriction is $v_b \in \bar{v}_1$. Therefore we can set $\bar{v}_1 = \{\bar{v}_{11}, v_b\}$ and develop $K_\ell(\bar{v}_1|\bar{u}_1)$ over the residues at $v_b = u_i$, where $u_i \in \bar{u}_1$. Let $\bar{u}_{11} = \bar{u}_1 \setminus \{u_i\}$. Then

$$K_\ell(\bar{v}_1|\bar{u}_1) = \sum g(v_b, u_i)f(\bar{v}_{11}, u_i)f(u_i, \bar{u}_{11})K_{\ell-1}(\bar{v}_{11}|\bar{u}_{11}), \quad (7.46)$$

where the sum is taken over partitions $\bar{u}_1 \Rightarrow \{\bar{u}_{11}, u_i\}$, and the subset u_i consists of one element. Substituting this into (7.45) we arrive at

$$\begin{aligned} M_1 - h(\bar{v}_b, v_b)Y_{a,b}(\bar{u}, \bar{v}) &= -h(\bar{v}_b, v_b) \sum g(v_b, u_i)f(\bar{v}_{11}, u_i)f(u_i, \bar{u}_{11})K_{\ell-1}(\bar{v}_{11}|\bar{u}_{11}) \\ &\times f(\bar{u}_{11}, \bar{u}_\Pi)f(u_i, \bar{u}_\Pi)g(\bar{v}_{\Pi'}, \bar{v}_{11})g(\bar{v}_{\Pi'}, v_b) \frac{T_{13}(v_b)\mathbb{T}_{13}(\bar{v}_{11})}{h(\bar{v}_{11}, v_b)} \mathbb{T}_{23}(\bar{v}_{\Pi'}) T_{12}(\bar{u}_\Pi)T_{22}(\bar{u}_{11})T_{22}(u_i). \end{aligned} \quad (7.47)$$

Here the sum is taken over partitions $\bar{v}_b \Rightarrow \{\bar{v}_{11}, \bar{v}_{\Pi'}\}$ and $\bar{u} \Rightarrow \{u_i, \bar{u}_{11}, \bar{u}_\Pi\}$ with the restrictions $\#\bar{u}_{11} = \#\bar{v}_{11} = \ell - 1$ and $\#u_i = 1$. Setting here $\bar{u}_0 = \bar{u} \setminus \{u_i\}$ we finally obtain

$$\begin{aligned} M_1 - h(\bar{v}_b, v_b)Y_{a,b}(\bar{u}, \bar{v}) &= -T_{13}(v_b) \sum g(v_b, u_i)f(\bar{v}_{11}, u_i)f(\bar{v}_{\Pi'}, v_b)f(u_i, \bar{u}_0) \\ &\times K_{\ell-1}(\bar{v}_{11}|\bar{u}_{11})f(\bar{u}_{11}, \bar{u}_\Pi)g(\bar{v}_{\Pi'}, \bar{v}_{11}) \mathbb{T}_{13}(\bar{v}_{11})\mathbb{T}_{23}(\bar{v}_{\Pi'}) T_{12}(\bar{u}_\Pi)T_{22}(\bar{u}_{11})T_{22}(u_i). \end{aligned} \quad (7.48)$$

Let us consider now the contribution M_2 :

$$M_2 = \sum K_\ell(\bar{v}_I|\bar{u}_I) f(\bar{u}_I, \bar{u}_\Pi) g(\bar{v}_\Pi, \bar{v}_{II}) g(v_i, v_b) g(v_i, \bar{v}_{II}) \mathbb{T}_{13}(\bar{v}_{II}) T_{23}(v_i) \mathbb{T}_{23}(\bar{v}_\Pi) T_{12}(\bar{u}_\Pi) T_{22}(\bar{u}_I), \quad (7.49)$$

Here the sum is taken over partitions $\bar{v}_b \Rightarrow \{\bar{v}_I, \bar{v}_\Pi\}$ and $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_\Pi\}$ with the restriction $\#\bar{u}_I = \#\bar{v}_I = \ell$, and then the subset \bar{v}_I is divided into v_i and \bar{v}_{II} , where v_i consists of one element. The goal is to combine v_i and \bar{v}_Π into subset $\bar{v}_{\Pi'}$. For this we first transform (7.49) as follows:

$$M_2 = \sum (-1)^\ell K_\ell(\{\bar{v}_{II}, v_i\}|\bar{u}_I) g(\bar{v}_0, v_i) f(\bar{u}_I, \bar{u}_\Pi) g(\bar{v}_\Pi, \bar{v}_{II}) \mathbb{T}_{13}(\bar{v}_{II}) T_{23}(v_i) \mathbb{T}_{23}(\bar{v}_\Pi) T_{12}(\bar{u}_\Pi) T_{22}(\bar{u}_I), \quad (7.50)$$

where $\bar{v}_0 = \bar{v} \setminus \{v_i\}$. Then we introduce $\bar{v}_{\Pi'} = \{v_i, \bar{v}_\Pi\}$ and obtain

$$M_2 = \sum (-1)^\ell K_\ell(\{\bar{v}_{II}, v_i\}|\bar{u}_I) \frac{g(\bar{v}_0, v_i)}{g(v_i, \bar{v}_{II})} f(\bar{u}_I, \bar{u}_\Pi) g(\bar{v}_{\Pi'}, \bar{v}_{II}) h(\bar{v}_{\Pi'}, v_i) \times \mathbb{T}_{13}(\bar{v}_{II}) \mathbb{T}_{23}(\bar{v}_{\Pi'}) T_{12}(\bar{u}_\Pi) T_{22}(\bar{u}_I). \quad (7.51)$$

Now we again develop $K_\ell(\{\bar{v}_{II}, v_i\}|\bar{u}_I)$ with respect to the poles at $v_i = u_i$, $u_i \in \bar{u}_I$. This gives us

$$M_2 = - \sum \left[g(v_i, u_i) g(v_b, v_i) f(\bar{v}_\Pi, v_i) \right] f(\bar{v}_{II}, u_i) f(u_i, \bar{u}_0) \times K_{\ell-1}(\bar{v}_{II}|\bar{u}_{II}) f(\bar{u}_{II}, \bar{u}_\Pi) g(\bar{v}_{\Pi'}, \bar{v}_{II}) \mathbb{T}_{13}(\bar{v}_{II}) \mathbb{T}_{23}(\bar{v}_{\Pi'}) T_{12}(\bar{u}_\Pi) T_{22}(\bar{u}_{II}) T_{22}(u_i). \quad (7.52)$$

Here the sum is taken over partitions $\bar{v}_b \Rightarrow \{v_i, \bar{v}_{II}, \bar{v}_\Pi\}$ and $\bar{u} \Rightarrow \{u_i, \bar{u}_{II}, \bar{u}_\Pi\}$ with the restrictions $\#\bar{u}_{II} = \#\bar{v}_{II} = \ell - 1$ and $\#u_i = \#v_i = 1$. Hereby $\bar{v}_{\Pi'} = \{v_i, \bar{v}_\Pi\}$ and we introduced $\bar{u}_0 = \bar{u} \setminus \{u_i\}$.

Now we can take the sum over v_i (see the terms in the square brackets in (7.52)). Recall that v_i runs through the subset $\bar{v}_{\Pi'}$. Therefore, we easily find by means of contour integration

$$\sum_{v_i \in \bar{v}_{\Pi'}} g(v_i, u_i) g(v_b, v_i) f(\bar{v}_\Pi, v_i) = g(v_b, u_i) \left\{ f(\bar{v}_{\Pi'}, u_i) - f(\bar{v}_{\Pi'}, v_b) \right\}. \quad (7.53)$$

Hence,

$$M_2 = - \sum f(\bar{v}_{II}, u_i) f(u_i, \bar{u}_0) g(v_b, u_i) \left\{ f(\bar{v}_{\Pi'}, u_i) - f(\bar{v}_{\Pi'}, v_b) \right\} \times K_{\ell-1}(\bar{v}_{II}|\bar{u}_{II}) f(\bar{u}_{II}, \bar{u}_\Pi) g(\bar{v}_{\Pi'}, \bar{v}_{II}) \mathbb{T}_{13}(\bar{v}_{II}) \mathbb{T}_{23}(\bar{v}_{\Pi'}) T_{12}(\bar{u}_\Pi) T_{22}(\bar{u}_{II}) T_{22}(u_i). \quad (7.54)$$

Finally, combining (7.54) and (7.48) we arrive at

$$T_{23}(v_b) Y_{a,b-1}(\bar{u}, \bar{v}_b) - h(\bar{v}_b, v_b) Y_{a,b}(\bar{u}, \bar{v}) = -T_{13}(v_b) \sum g(v_b, u_i) f(\bar{v}_b, u_i) f(u_i, \bar{u}_0) \times \left\{ K_{\ell-1}(\bar{v}_{II}|\bar{u}_{II}) f(\bar{u}_{II}, \bar{u}_\Pi) g(\bar{v}_{\Pi'}, \bar{v}_{II}) \mathbb{T}_{13}(\bar{v}_{II}) \mathbb{T}_{23}(\bar{v}_{\Pi'}) T_{12}(\bar{u}_\Pi) T_{22}(\bar{u}_{II}) \right\} T_{22}(u_i). \quad (7.55)$$

The sum in the r.h.s. is organized as follows. First we divide the set \bar{u} into subset u_i (with $\#u_i = 1$) and the complementary subset \bar{u}_0 . Then the have additional partitions $\bar{u}_0 \Rightarrow \{\bar{u}_{II}, \bar{u}_\Pi\}$

and $\bar{v}_b \Rightarrow \{\bar{v}_{\text{ii}}, \bar{v}_{\text{ii}'}\}$ with $\#\bar{u}_{\text{ii}} = \#\bar{v}_{\text{ii}} = \ell - 1$ (see the terms in the braces in the second line of (7.55)). Clearly the sum over these partitions gives $Y_{a-1,b-1}(\bar{u}_0, \bar{v}_b)$, and we arrive at

$$\begin{aligned} T_{23}(v_b)Y_{a,b-1}(\bar{u}, \bar{v}_b) - h(\bar{v}_b, v_b)Y_{a,b}(\bar{u}, \bar{v}) \\ = - \sum g(v_b, u_i)f(\bar{v}_b, u_i)f(u_i, \bar{u}_0)T_{13}(v_b)Y_{a-1,b-1}(\bar{u}_0, \bar{v}_b)T_{22}(u_i), \end{aligned} \quad (7.56)$$

which gives us the recursion (7.9).

8 Expressions for $Y(1|2)$ Bethe vectors and dual Bethe vectors

Once we have explicit expressions for Bethe vectors in $Y(2|1)$, the morphisms φ , ψ and $\tilde{\psi}$ allow us to get explicit expressions for the remaining (dual) Bethe vectors, as detailed in the following propositions.

Proposition 8.1. *In $Y(1|2)$ the Bethe vectors have the two following explicit expressions*

$$\tilde{\Phi}_{a,b}(\bar{u}, \bar{v}) = \sum (-1)^b g(\bar{u}_{\text{I}}, \bar{v}_{\text{I}}) f(\bar{v}_{\text{I}}, \bar{v}_{\text{II}}) g(\bar{u}_{\text{II}}, \bar{u}_{\text{I}}) h(\bar{v}_{\text{I}}, \bar{v}_{\text{I}}) \tilde{\mathbb{T}}_{13}(\bar{v}_{\text{I}}) \tilde{T}_{23}(\bar{v}_{\text{II}}) \tilde{\mathbb{T}}_{12}(\bar{u}_{\text{II}}) \tilde{\lambda}_2(\bar{u}_{\text{I}}) \tilde{\Omega}, \quad (8.1)$$

$$\tilde{\Phi}_{a,b}(\bar{u}, \bar{v}) = \sum (-1)^b K_{\ell}(\bar{u}_{\text{I}}|\bar{v}_{\text{I}}) f(\bar{v}_{\text{I}}, \bar{v}_{\text{II}}) g(\bar{u}_{\text{II}}, \bar{u}_{\text{I}}) \tilde{\mathbb{T}}_{13}(\bar{u}_{\text{I}}) \tilde{\mathbb{T}}_{12}(\bar{u}_{\text{II}}) \tilde{T}_{23}(\bar{v}_{\text{II}}) \tilde{\lambda}_2(\bar{v}_{\text{I}}) \tilde{\Omega}. \quad (8.2)$$

In relations (8.1) and (8.2), the sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_{\text{I}}, \bar{v}_{\text{II}}\}$ and $\bar{u} \Rightarrow \{\bar{u}_{\text{I}}, \bar{u}_{\text{II}}\}$ with the restriction $\#\bar{u}_{\text{I}} = \#\bar{v}_{\text{I}} = \ell$, $0 \leq \ell \leq \min(a, b)$. We have also introduced (for $\#\bar{z} = k$)

$$\tilde{\mathbb{T}}_{12}(\bar{z}) = \frac{1}{H(\bar{z})} \prod_{1 \leq j \leq k}^{\rightarrow} \tilde{T}_{12}(z_j) \quad \text{and} \quad \tilde{\mathbb{T}}_{13}(\bar{z}) = \frac{1}{H(\bar{z})} \prod_{1 \leq j \leq k}^{\rightarrow} \tilde{T}_{13}(z_j). \quad (8.3)$$

Proof: We obtain the relations by application of φ and $\tilde{\varphi}$ to the relations (7.1) and (7.2). The proof is similar to the ones of section 5. \blacksquare

Proposition 8.2. *In $Y(2|1)$, the dual Bethe vectors $\Psi_{ab}(\bar{u}, \bar{v})$ comply the following explicit expressions:*

$$\Psi_{a,b}(\bar{u}, \bar{v}) = \Omega^{\dagger} (-1)^{(b-1)b/2} \sum g(\bar{v}_{\text{I}}, \bar{u}_{\text{I}}) f(\bar{u}_{\text{I}}, \bar{u}_{\text{II}}) g(\bar{v}_{\text{II}}, \bar{v}_{\text{I}}) h(\bar{u}_{\text{I}}, \bar{u}_{\text{I}}) \lambda_2(\bar{v}_{\text{I}}) \mathbb{T}_{32}(\bar{v}_{\text{II}}) T_{21}(\bar{u}_{\text{II}}) \mathbb{T}_{31}(\bar{u}_{\text{I}}), \quad (8.4)$$

$$\Psi_{a,b}(\bar{u}, \bar{v}) = \Omega^{\dagger} (-1)^{(b-1)b/2} \sum K_{\ell}(\bar{v}_{\text{I}}|\bar{u}_{\text{I}}) f(\bar{u}_{\text{I}}, \bar{u}_{\text{II}}) g(\bar{v}_{\text{II}}, \bar{v}_{\text{I}}) \lambda_2(\bar{u}_{\text{I}}) T_{21}(\bar{u}_{\text{II}}) \mathbb{T}_{32}(\bar{v}_{\text{II}}) \mathbb{T}_{31}(\bar{v}_{\text{I}}), \quad (8.5)$$

In $Y(1|2)$, the dual Bethe vectors $\tilde{\Psi}_{ab}(\bar{u}, \bar{v})$ have the following explicit expressions:

$$\begin{aligned} \tilde{\Psi}_{a,b}(\bar{u}, \bar{v}) &= \tilde{\Omega}^{\dagger} (-1)^{b+(a-1)a/2} \sum g(\bar{u}_{\text{I}}, \bar{v}_{\text{I}}) f(\bar{v}_{\text{I}}, \bar{v}_{\text{II}}) g(\bar{u}_{\text{II}}, \bar{u}_{\text{I}}) h(\bar{v}_{\text{I}}, \bar{v}_{\text{I}}) \tilde{\lambda}_2(\bar{u}_{\text{I}}) \\ &\quad \times \tilde{\mathbb{T}}_{21}(\bar{u}_{\text{II}}) \tilde{T}_{32}(\bar{v}_{\text{II}}) \tilde{\mathbb{T}}_{31}(\bar{v}_{\text{I}}) \end{aligned} \quad (8.6)$$

$$\begin{aligned} \tilde{\Psi}_{a,b}(\bar{u}, \bar{v}) &= \tilde{\Omega}^{\dagger} (-1)^{b+(a-1)a/2} \sum K_{\ell}(\bar{u}_{\text{I}}|\bar{v}_{\text{I}}) f(\bar{v}_{\text{I}}, \bar{v}_{\text{II}}) g(\bar{u}_{\text{II}}, \bar{u}_{\text{I}}) \tilde{\lambda}_2(\bar{v}_{\text{I}}) \\ &\quad \times \tilde{T}_{32}(\bar{v}_{\text{II}}) \tilde{\mathbb{T}}_{21}(\bar{u}_{\text{II}}) \tilde{\mathbb{T}}_{31}(\bar{u}_{\text{I}}). \end{aligned} \quad (8.7)$$

Again, the sums are taken over partitions $\bar{v} \Rightarrow \{\bar{v}_I, \bar{v}_{II}\}$ and $\bar{u} \Rightarrow \{\bar{u}_I, \bar{u}_{II}\}$ with the restriction $0 \leq \#\bar{u}_I = \#\bar{v}_I = \ell \leq \min(a, b)$. We have introduced

$$\mathbb{T}_{32}(\bar{v}) = \frac{1}{H(\bar{v}^*)} \prod_{1 \leq j \leq \ell}^{\rightarrow} T_{32}(v_j) \quad , \quad \mathbb{T}_{31}(\bar{v}) = \frac{1}{H(\bar{v}^*)} \prod_{1 \leq j \leq \ell}^{\rightarrow} T_{31}(v_j), \quad (8.8)$$

$$\tilde{\mathbb{T}}_{21}(\bar{u}) = \frac{1}{H(\bar{u}^*)} \prod_{1 \leq j \leq \ell}^{\rightarrow} \tilde{T}_{21}(u_j) \quad \text{and} \quad \tilde{\mathbb{T}}_{31}(\bar{u}) = \frac{1}{H(\bar{u}^*)} \prod_{1 \leq j \leq \ell}^{\rightarrow} \tilde{T}_{31}(u_j). \quad (8.9)$$

Proof: To get (8.4) and (8.5), we apply the antimorphism ψ to the explicit expressions of BVs (7.1) and (7.2). Expressions (8.6) and (8.7) are obtained from (8.1) and (8.2) with the use of $\tilde{\psi}$. The proof is similar to the ones of section 6. In particular, it makes appear $H(\bar{v}^*)$ for the definition of e.g. $\mathbb{T}_{32}(\bar{v})$, as in the super-trace formula for dual Bethe vectors. ■

Acknowledgements

Work of S.P. was supported in part by RFBR grant 16-01-00562. N.A.S. was supported by grants RFBR-15-31-20484-mol-a-ved, RFBR-14-01-00860-a.

References

- [1] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, *Quantum Inverse Problem. I*, Theor. Math. Phys. **40** (1979) 688–706;
L.D. Faddeev and L.A. Takhtajan, *The quantum method of the inverse problem and the Heisenberg XYZ model*, Usp. Math. Nauk **34** (1979) 13–63; Russian Math. Surveys **34** (1979) 11–68 (Engl. transl.).
- [2] V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge: Cambridge Univ. Press, 1993.
- [3] L.D. Faddeev, in: *Les Houches Lectures Quantum Symmetries*, eds A. Connes et al, North Holland, (1998) 149.
- [4] A.G. Izergin, V.E. Korepin, *The quantum inverse scattering method approach to correlation functions*, Comm. Math. Phys. **94** (1984) 67–92.
- [5] V.E. Korepin, *Calculation of norms of Bethe wave functions*, Comm. Math. Phys. **86** (1982) 391–418;
A.G. Izergin, V.E. Korepin, *The quantum inverse scattering method approach to correlation functions*, Comm. Math. Phys. **94** (1984), 67–92.
- [6] N.A. Slavnov, *Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz*, Theor. Math. Phys. **79** (1989) 502–508.

- [7] N. Kitanine, K.K. Kozłowski, J.M. Maillet, N.A. Slavnov, V. Terras, *Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions*, J. Stat. Mech.: Theory and Exp. **04** (2009) P04003, [arXiv:0808.0227](#).
- [8] M. Dugave, F. Goehmann, K.K. Kozłowski, *Low-temperature large-distance asymptotics of the transversal two-point functions of the XXZ chain*, J. Stat. Mech. (2014) P04012, [arXiv:1401.4132](#).
- [9] P.P. Kulish, N.Yu. Reshetikhin, *GL(3)-invariant solutions of the Yang-Baxter equation and associated quantum systems*, Zap. Nauchn. Sem. POMI. **120** (1982) 92–121; J. Sov. Math. **34:5** (1982) 1948–1971 (Engl. transl.).
- [10] N.Yu. Reshetikhin, *Calculation of the norm of Bethe vectors in models with SU(3)-symmetry*, Zap. Nauchn. Sem. LOMI **150** (1986) 196–213; J. Math. Sci. **46** (1989) 1694–1706 (Engl. transl.).
- [11] S. Belliard, S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Bethe vectors of GL(3)-invariant integrable models*, J. Stat. Mech. **1302** (2013) P02020, [arXiv:1210.0768](#).
- [12] S. Belliard, S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Bethe vectors of quantum integrable models with GL(3) trigonometric R-matrix*, SIGMA **9** (2013) 058, [arXiv:1304.7602](#).
- [13] S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Scalar products in models with GL(3) trigonometric R-matrix. Highest coefficient*, Theor. Math. Phys. **178** (2014) 314–335, [arXiv:1311.3500](#); S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Scalar products in models with \mathfrak{gl}_3 trigonometric R-matrix. General case*, Theor. Math. Phys. **180:1** (2014) 795–814, [arXiv:1401.4355](#).
- [14] N.A. Slavnov, *Scalar products in GL(3)-based models with trigonometric R-matrix. Determinant representation*, J. Stat. Mech. Theory Exp. **1503** (2015) P03019, [arXiv:1501.06253](#).
- [15] S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Determinant representations for form factors in quantum integrable models with GL(3)-invariant R-matrix*, Theor. Math. Phys. **181** (2014) 1566–1584, [arXiv:1406.5125](#);
S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Form factors in quantum integrable models with GL(3)-invariant R-matrix*, Nucl. Phys. **B881** (2014) 343–368, [arXiv:1312.1488](#);
S. Belliard, S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Form factors in SU(3)-invariant integrable models*, J. Stat. Mech. **1309** (2013) P04033, [arXiv:1211.3968](#).
- [16] S. Pakuliak, E. Ragoucy, N.A. Slavnov, *GL(3)-based quantum integrable composite models: 1. Bethe vectors*, SIGMA **11** (2015) 063, [arXiv:1501.07566](#).
- [17] N.A. Slavnov, *One-dimensional two-component Bose gas and the algebraic Bethe ansatz*, Theor. Math. Phys. **183:3** (2015) 800–821, [arXiv:1502.06749](#).
- [18] B. Pozsgay, W.-V. van G. Oei and M. Kormos, *On Form Factors in nested Bethe Ansatz systems*, J. Phys. A: Math. Gen. **45** (2012) 465007, [arXiv:1204.4037](#)

- [19] S. Pakuliak, E. Ragoucy, N.A. Slavnov, *GL(3)-based quantum integrable composite models: 2. Form factors of local operators*, SIGMA **11** (2015) 064, [arXiv:1502.01966](#).
- [20] S. Pakuliak, E. Ragoucy, N.A. Slavnov, *Form factors of local operators in a one-dimensional two-component Bose gas*, J. Phys. **A48** (2015) 435001, [arXiv:1503.00546](#).
- [21] B. Enriquez, S. Khoroshkin, S. Pakuliak, *Weight functions and Drinfeld currents*, Comm. Math. Phys. **276** (2007) 691–725, [arXiv:math/0610398](#);
S. Khoroshkin, S. Pakuliak, *Weight function for $\mathcal{U}_q(\widehat{\mathfrak{sl}}_3)$* , Theor. Math. Phys. **145** (2005) 1373–1399, [arXiv:math/0610433](#);
S. Khoroshkin, S. Pakuliak, *A computation of an universal weight function for the quantum affine algebra $\mathcal{U}_q(\mathfrak{gl}_N)$* , J. Math. of Kyoto University **48** (2008) 277–321, [arXiv:0711.2819](#);
A. Os’kin, S. Pakuliak, A. Silantyev, *On the universal weight function for the quantum affine algebra $\mathcal{U}_q(\mathfrak{gl}_N)$* , Algebra and Analysis **21** (2009) 196–240, [arXiv:0711.2821](#);
L. Frappat, S. Khoroshkin, S. Pakuliak, E. Ragoucy, *Bethe Ansatz for the Universal Weight Function*, Ann. H. Poincaré **10** (2009) 513–548, [arXiv:0810.3135](#);
S. Belliard, S. Pakuliak, E. Ragoucy, *Bethe Ansatz and Bethe Vectors Scalar Products*, SIGMA **6** (2010) 094, [arXiv:1012.1455](#).
- [22] V. Tarasov and A. Varchenko, *Combinatorial formulae for nested Bethe vector*, SIGMA **9** (2013) 048, [arXiv:math.QA/0702277](#).
- [23] S. Belliard and É. Ragoucy, *The nested Bethe ansatz for ‘all’ closed spin chains*, J. Phys. **A41** (2008) 295202, [arXiv:math-ph/0804.2822](#).
- [24] O. Foda, M. Wheeler, *Variations on Slavnov’s scalar product*, JHEP **1210** (2012) 096, [arXiv:1207.6871](#).
- [25] C. Ahn, O. Foda, R.I. Nepomechie, *OPE in planar QCD from integrability*, JHEP **1206** (2012) 168, [arXiv:1202.6553](#).
- [26] J.M. Drummond, E. Ragoucy, *Superstring amplitudes and the associator*, JHEP **1308** (2013) 135, [arXiv:1301.0794](#).
- [27] N. Beisert et. al., *Review of AdS/CFT Integrability: An Overview*, Lett. Math. Phys. **99** (2012) 3–32, [arXiv:1012.3982](#).
- [28] N. Beisert, M. Staudacher, *Long-Range PSU(2,2|4) Bethe Ansatz for Gauge Theory and Strings*, Nucl. Phys. **B727** (2005) 1–62, [arXiv:hep-th/0504190](#).
- [29] N. Beisert, B.I. Zwiebel, *On Symmetry Enhancement in the $psu(1,1|2)$ Sector of N=4 SYM*, JHEP **0710** (2007) 031, [arXiv:0707.1031](#).
- [30] S. Sarkar, *The supersymmetric t-J model in one dimension*, J. Physics **A24** (1991) 1137–1152;
S. Sarkar, *Bethe-ansatz solution of the t-J model*, J. Physics **A23** (1990) L409–L414;
P.A. Bares, G. Blatter, M. Ogata, *Exact solution of the t-J model in one dimension at $2t = \pm J$: Ground state and excitation spectrum*, Phys. Rev. **B44** (1991) 130–154;

F.H.L. Essler, V.E. Korepin, *Higher conservation laws and algebraic Bethe ansatz for the supersymmetric t - J model*, Phys. Rev. **B46** (1992) 9147–9162 .